bility of factors of the forms

$$2^t \cdot 3 + 1, \quad 2^t \cdot 5 + 1, \quad 2^t \cdot 7 + 1.$$

In particular, a prime of the form $2^{2^t} \cdot 3 + 1$ cannot be a factor of a Fermat number F_n.[1] The smallest of the as yet known factors of each F_n is actually the least factor of each such number.

Again it was noticed early that no two F_n have a common factor. For, if F_n and F_m, $m > n$, have a common prime factor p, then

$$2^{2^m} \equiv (2^{2^n})^{2^{m-n}} \equiv (-1)^{2^{m-n}} \equiv 1 \bmod p,$$

so that $2^{2^m} + 1$ is not divisible by p.

7. The first theorem about the numbers F_n actually demonstrated is that of Euler to the effect that all factors of F_n are of the form $2^{n+1} \cdot k + 1$. We give a proof of Lucas' extension of this theorem, namely,

Every factor of a given number F_n ($n \geqq 2$) is of the form $2^{n+2} \cdot k + 1$.

It is sufficient to prove the theorem for a prime factor p of F_n. Since

$$2^{2^n} \equiv -1 \bmod p$$

it is evident that 2^{n+1} is the exponent to which 2 belongs modulo p, so that $p - 1$ is divisible by 2^{n+1} (the result due to Euler). Now, p is of the form $8t + 1$, since $n \geqq 2$. Hence 2 is a quadratic residue modulo p. Therefore, we have

$$2^{\frac{1}{2}(p-1)} \equiv 1 \bmod p,$$

so that $\frac{1}{2}(p - 1)$ is divisible by 2^{n+1}, the exponent to which 2 belongs modulo p. From this the theorem follows at once.

By means of this theorem it is easy to establish the following proposition:

If $f = 2^{n+2} \cdot q + 1$ is a factor of F_n and $q \not> 9 \cdot 2^{n+2}$, then f is a prime.

For if f is not prime we must have

$$f = (2^{n+2} \cdot x + 1)(2^{n+2} \cdot y + 1)$$

where x and y are integers greater than 2; whence we have $q > 9 \cdot 2^{n+2}$. [In this theorem 9 may be replaced by 15 if f is not a square number.]

For the numbers F_5, F_{12}, F_{23}, F_{36}, F_{38}, F_{73}, the least factors to be tried (in accordance with the foregoing theorem of Lucas and the fact that no two F_n have a common factor) are respectively

$$2^7 \cdot 5 + 1, \quad 2^{14} \cdot 7 + 1, \quad 2^{25} \cdot 5 + 1, \quad 2^{39} \cdot 5 + 1, \quad 2^{41} \cdot 3 + 1, \quad 2^{75} \cdot 5 + 1;$$

and in each case such number is a factor of the corresponding F_n. Thus of the seventeen cases in which we know at present whether the given F_n is prime or composite there are five in each of which it is easily seen to be prime while in half (six in number) of the remaining cases the least number to be tried in each instance turns out to be a factor. The verification by means of congruences is

[1] *Bull. Amer. Math. Soc.*, (2), 12: 450.

easy in the first three cases and not unduly tedious in the last three cases. In view of such verification and the two theorems above it is easy to see that each of these six factors is a prime number, the last being one of twenty-four digits.

8. Since the totient $\varphi(p)$ of p is less than $p - 1$ except when p is a prime number, the following theorem of Lucas is an immediate consequence of the Fermat theorem $a^{\phi(m)} \equiv 1 \bmod m$:

If the congruence $a^x \equiv 1 \bmod p$ is true when $x = p - 1$ but is not true when x is any proper divisor of $p - 1$, then p is a prime number.

For, $a^\delta \equiv 1 \bmod p$, when δ is the greatest common divisor of $\varphi(p)$ and $p - 1$.

By means of this result it is easy to establish Pepin's theorem concerning prime numbers F_n:

For $n > 1$, F_n is prime if and only if it divides

$$k^{\frac{1}{2}(F_n-1)} + 1,$$

where k is any quadratic non-residue of F_n, as 3 or 5 or 10.

A test of this sort for prime numbers is remarkable in that it depends not directly on seeking factors of the number in consideration but in ascertaining whether it divides some other number.

To prove Pepin's theorem we proceed as follows: If F_n is prime we have

$$k^{\frac{1}{2}(F_n-1)} \equiv -1 \bmod F_n$$

by the theory of quadratic residues. On the other hand, if the foregoing congruence is satisfied, we have

$$k^{F_n-1} \equiv 1 \bmod F_n$$

while there is no proper divisor d of $F_n - 1$ for which $k^d \equiv 1 \bmod F_n$, since every such divisor is a power of 2 and a factor of $\frac{1}{2}(F_n - 1)$. Applying the theorem of Lucas at the beginning of this section we now conclude to the truth of Pepin's theorem.

To apply Pepin's test, one would take consecutively the minimum residues modulo F_n of

$$k^2, k^4, k^8, \cdots, k^{2^{2^n-1}}.$$

9. Hurwitz gave the following generalization of Pepin's theorem:

Let $F_n(x)$ denote the irreducible algebraic factor of $x^n - 1$ (of degree $\varphi(n)$) which is not a factor of any $x^v - 1$ for which $v < n$. Then if there exists an integer q such that $F_{p-1}(q)$ is divisible by p, p is a prime.

When $p = 2^k + 1$ it is easy to show that $F_{p-1}(x) = x^{2^{k-1}} + 1$.

Carmichael employed the notation $F_n(\alpha, \beta)$ for $\beta^{\phi(n)} F_n(\alpha/\beta)$, where $F_n(x)$ has the meaning just defined. He gave the following generalization of the foregoing theorem of Hurwitz:

A necessary and sufficient condition that a given odd number p is prime is that there exist relatively prime integers $\alpha + \beta$ and $\alpha\beta$, α and β not both roots of unity, such that (the integer) $F_{p-1}(\alpha, \beta)$ is divisible by p.

A proof of this theorem is to be found in *Annals of Math.*, (2), 15 (1913): 66.

10. Let $\alpha + \beta$ and $\alpha\beta$ be two relatively prime integers, α and β not both roots of unity, and consider the quantities

$$S_n = \alpha^n + \beta^n, \qquad D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}.$$

S_n and D_n both represent integers, since each of them is a symmetric polynomial (with integral coefficients) in the roots α and β of the equation

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

whose coefficients are integers. It is obvious that D_n is a factor of D_{nk} if n and k are both integers.

Let p be an odd prime which is not a divisor of either $(\alpha - \beta)^2$ or $\alpha\beta$. We have

$$S_p - (\alpha + \beta) \equiv \alpha^p + \beta^p - (\alpha + \beta)^p \equiv 0 \bmod p$$

and

$$D_p - (\alpha - \beta)^{p-1} \equiv \alpha^{p-1} + \cdots + \beta^{p-1} - (\alpha - \beta)^{p-1} \equiv 0 \bmod p.$$

But it is easy to verify that

$$(\alpha + \beta)D_p - S_p = 2\alpha\beta D_{p-1}.$$

Therefore

$$(\alpha + \beta)(\alpha - \beta)^{p-1} - (\alpha + \beta) \equiv 2\alpha\beta D_{p-1} \bmod p.$$

Now $(\alpha - \beta)^2$ is an integer; hence from Fermat's theorem it follows that

$$(\alpha - \beta)^{p-1} \equiv \pm 1 \bmod p.$$

From the last two congruences we have therefore

$$D_{p-1} \equiv 0 \bmod p \qquad \text{if } (\alpha - \beta)^{p-1} \equiv 1 \bmod p,$$

$$\alpha\beta D_{p-1} \equiv -(\alpha + \beta) \bmod p \qquad \text{if } (\alpha - \beta)^{p-1} \equiv -1 \bmod p.$$

Now it is easy to verify that

$$D_{p+1} - (\alpha + \beta)D_p + \alpha\beta D_{p-1} = 0;$$

and hence we see that

$$D_{p+1} \equiv 0 \bmod p \qquad \text{if } (\alpha - \beta)^{p-1} \equiv -1 \bmod p.$$

Therefore we have the following theorem (due to Lucas):

An odd prime p which does not divide either $(\alpha - \beta)^2$ or $\alpha\beta$ is a factor of D_{p-1} or of D_{p+1} according as $(\alpha - \beta)^{p-1}$ is congruent to $+1$ or to -1 modulo p.

That D_n and S_n are both prime to $\alpha\beta$ follows at once from the readily verified relations

$$(\alpha + \beta)^n = \alpha^n + \beta^n + \alpha\beta I_1 = S_n + \alpha\beta I_1,$$

$$D_n = \alpha^{n-1} + \beta^{n-1} + \alpha\beta I_2 = S_{n-1} + \alpha\beta I_2,$$

in which I_1 and I_2 denote integers. But it is easy to show directly that

$$D_m S_n - D_n S_m = 2\alpha^n \beta^n D_{m-n}.$$

Hence a common odd prime factor p of D_m and D_n $(m > n)$ is a factor of D_{m-n}; whence it follows without difficulty that p is a factor of D_v where v is the greatest common divisor of m and n. From this conclusion and the foregoing theorem we see that the least value k of n for which D_n is divisible by a given prime p (not dividing $(\alpha - \beta)^2$ or $\alpha\beta$) is a factor of $p - 1$ or $p + 1$ in the respective cases. Hence we have the following theorem (also due to Lucas):

If D_{q+1} $[D_{q-1}]$ is divisible by the odd number q, q being prime to $(\alpha - \beta)^2$, but D_k is not divisible by q for any factor k of $q + 1$ $[q - 1]$, then q is a prime number.

From the easily verified relation

$$S_n^2 - (\alpha - \beta)^2 D_n^2 = 4\alpha^n \beta^n$$

and the fact that D_n and S_n are prime to $\alpha\beta$ it follows that D_n and S_n have no common odd prime factor. But

$$D_{2^k} = S_{2^{k-1}} D_{2^{k-1}} = S_{2^{k-1}} S_{2^{k-2}} \cdots S_{2^2} S_2 S_1.$$

Hence no two numbers in the sequence

$$S_{2^0},\ S_{2^1},\ S_{2^2},\ S_{2^3},\ \cdots$$

have a common odd prime factor.

11. For the application of these results to Fermat numbers F_n let us take $\alpha + \beta = 2$, $\alpha\beta = -1$. Then $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 8$, a quadratic residue of a prime number F_n, since 2 is a quadratic residue of all primes of the form $8t + 1$. All the divisors of $F_n - 1$ are powers of 2. Moreover

$$D_{2^k} = S_{2^{k-1}} S_{2^{k-2}} \cdots S_2 S_1.$$

Hence we are concerned primarily with the numbers $G_1 = 6$, G_2, G_3, $\cdots$, where

$$G_s = S_{2^s}.$$

It is easy to verify directly the recurrence relation

$$G_{s+1} = G_s^2 - 2, \qquad s > 0.$$

Moreover no two G's have a common odd prime factor, as we see from the statement at the close of § 10.

From the first theorem of the preceding section it follows that $D_{F_n - 1}$ is divisible by F_n if F_n is prime. Hence F_n is composite if it divides no one of the numbers $G_1, G_2, \cdots G_{2^{n-1}}$.

Suppose that F_n divides G_t, $t < 2^n$, but no G with subscript less than t. Then any prime factor p of F_n divides G_t but no G with subscript less than t, since the G's are relatively prime in pairs. Hence p is a factor of $D_{2^{t+1}}$ but of no

D_s where $s < 2^{t+1}$. Hence p is of the form $2^{t+1} \cdot q + 1$. Hence it follows that F_n is prime if $2^{n-1} \leqq t \leqq 2^n - 1$; for, if F_n is not prime, we should have

$$F_n = (2^{t+1} \cdot q_1 + 1)(2^{t+1} \cdot q_2 + 1) > 2^{2(t+1)} + 1;$$

or $2(t+1) < 2^n$.

These results yield us the following remarkable theorem of Lucas:

Consider the sequence $G_1 = 6$, $G_2 = 34$, $G_3 = 1154$, G_4, $\cdots$, G_{2^n-1}, *each term of which is* 2 *less than the square of the preceding. Then* F_n *is composite if it is a factor of no number of this sequence; if the first number of the sequence which* F_n *divides is* G_t, *then every prime factor of* F_n *is of the form* $2^{t+1} \cdot q + 1$, *and* F_n *is a prime of this form if* $t \geqq 2^{n-1}$.

Other Known Properties of the Numbers F_n.

12. A few other properties of the numbers F_n should be mentioned.[1]

J. Hermes indicated a test for composite F_n different from those given in the foregoing sections.

H. Scheffler noted that

$$F_n F_{n+1} \cdots F_{a-1} = 1 + 2^{2^n} + 2^{2 \cdot 2^n} + 2^{3 \cdot 2^n} + \cdots + 2^{2^a - 2^n}.$$

A. Cunningham considered the period of $1/N$ to base 2, where $N = F_m F_{m-1} \cdots F_{m-r}$. He noted that every $F_n > 5$ can be represented by four quadratic forms of determinants $\pm G_n$, $\pm 2G_n$, where $G_n = F_0 F_1 \cdots F_{n-1}$.

Gosset gave the complex prime factors $a \pm b\sqrt{-1}$ of the known real prime factors of composite F_n for $n = 5, 6, 9, 11, 12, 18, 23, 36, 38$.

Cunningham gave several algebraic properties of the numbers F_n (listed by Dickson at references 53, 55 and 56).

It is believed that all the essential known properties of the numbers F_n are stated or referred to explicitly in the foregoing portions of this paper.

Conjectures and Outstanding Problems Mentioned in the Literature.

13. Lucas[2] has listed five important problems in the theory of prime numbers, arranged in the "probable order of difficulty": (1) To find a prime greater than a given prime; (2) To find a function which yields only prime numbers; (3) To find the prime which follows a given prime; (4) To find the number of primes below a given limit; (5) To calculate directly the prime number of given rank. The first two of these would be simultaneously solved if one could prove the following conjecture:

All the numbers of the sequence

$$2^2 + 1, \quad 2^{2^2} + 1, \quad 2^{2^{2^2}} + 1, \quad \cdots$$

are primes. (The prime $2^8 + 1$ does not belong to this sequence.)

[1] See Dickson, *loc. cit.*, for references.

[2] *Théorie des nombres*, I, p. 354.

A proof of the foregoing conjecture would also carry with it the establishment of a theorem stated by Eisenstein, namely, that there is an infinitude of primes F_n. No proof of this theorem has yet been given.

Lipschitz[1] separated all integers into classes, the primes of one class being Fermat numbers, and gave a new interest to the question of the infinitude of primes F_n.

Minor Additional Results.

14. Let p be a prime proper factor of F_n and write

$$p = 2^t \cdot k + 1$$

where k may or may not contain the factor 2 but is not a power of 2. A proper prime factor of F_n we shall call a Fermat prime factor.

Let us consider first the case in which $t = 2^\alpha$. (It is clear that 2^α may, if desired, be taken not less than the highest power of 2 not exceeding $n + 2$.) From the congruences

$$2^{2^n} \equiv -1, \qquad 2^{2^\alpha} \cdot k \equiv -1 \bmod p$$

it is clear that we have

$$k^{2^{n-\alpha}} + 1 \equiv 0 \bmod p. \tag{1}$$

From this we have readily the following theorem:

Every Fermat prime factor p has the property that it is a factor of a number of the form

$$a^{2^s} + 1$$

in which a is a factor of $p - 1$ belonging modulo p to the exponent 2^{s+1} and is not a power of 2.

The exponent s need not be taken greater than $n - \alpha$ where p is a factor of F_n and $2^\alpha \leqq n - 2$.

For the case of the prime factor p, $p = 2^{2^\alpha} \cdot k + 1$, of F_n it is obvious that $\alpha \leqq n - 1$. Consider the possibility that $\alpha = n - 1$. Then from (1) it follows that $k^2 + 1$ is divisible by p and hence that $k > 2^{2^{n-1}}$; whence we have a contradiction. Therefore, $\alpha \leqq n - 2$.

We have $k^4 + 1$ divisible by p when $\alpha = n - 2$; in view of (1) it is clear that it cannot be divisible by p when $\alpha < n - 2$. Hence the Fermat prime factors p of the form

$$p = 2^{2^{n-2}} \cdot k + 1$$

are all divisors of $k^4 + 1$ and no other Fermat prime factor $2^{2^\alpha} \cdot k + 1$ is a divisor of $k^4 + 1$.

In general, the Fermat prime factors of the form

$$2^{2^{n-\nu}} \cdot k + 1$$

[1] *Journ. für Math.*, 105, 152–156.

are divisors of $k^{2^\nu} + 1$ and no other Fermat prime factor $2^{2^n} \cdot k + 1$ is a divisor of $k^{2^\nu} + 1$.

15. In the Fermat prime factor p of F_n,

$$p = 2^t \cdot k + 1,$$

let t be an odd number. Then integers r and s exist such that

$$r \cdot 2^n - s \cdot t = 1.$$

From the congruences

$$2^{2^n} \equiv -1, \qquad 2^t \cdot k \equiv -1 \bmod p \tag{2}$$

we have

$$2^{r \cdot 2^n} \equiv (-1)^r, \qquad 2^{st} \cdot k^s \equiv (-1)^s \bmod p.$$

Hence we have

$$k^s \equiv (-1)^{r+s} \cdot 2 \bmod p.$$

It is clear that s is odd. Hence we have

$$k^s \equiv (-1)^{r+1} \cdot 2 \bmod p. \tag{3}$$

Similarly, integers r_1 and s_1 exist such that

$$s_1 t - r_1 2^n = 1,$$

and we have readily

$$2k^{s_1} \equiv (-1)^{r_1+1} \bmod p. \tag{4}$$

If we multiply (3) through by 2 and compare the result with (4) we see that

$$k^{s-s_1} \equiv (-1)^{r+r_1} \cdot 4 \qquad \text{or} \qquad 4k^{s_1-s} \equiv (-1)^{r+r_1} \bmod p \tag{5}$$

according as $s > s_1$ or $s < s_1$.

Again, from (2) we have immediately that

$$k^{2^n} + 1 \equiv 0 \bmod p \tag{6}$$

so that $2^{2^n} + 1$ and $k^{2^n} + 1$ have the common factor p.

From the two equations

$$r2^n - st = 1, \qquad s_1 t - r_1 2^n = 1 \tag{7}$$

we see that

$$2^n(r + r_1) = t(s + s_1). \tag{8}$$

If r, s; r_1, s_1 are the least positive integers satisfying (7) it is clear that s and s_1 are both less than 2^n. Hence from (8) we see that

$$s + s_1 = 2^n, \qquad r + r_1 = t. \tag{9}$$

Then equations (5) are respectively

$$k^{s-s_1} + 4 \equiv 0, \qquad 4k^{s_1-s} + 1 \equiv 0 \bmod p. \tag{10}$$

16. By means of (1) and the known factorizations of F_n one may readily obtain one numerical factor each of several binomial expressions. These are

given in the following table which contains in its first column a number F_n, in its second column a prime factor of F_n and in its third column a binomial number having the same factor, the table being complete as to all known prime factors of composite F_n:

F_5	$2^7 \cdot 5 + 1$	$40^8 + 1$
F_5	$2^7 \cdot 52347 + 1$	$418776^8 + 1$
F_6	$2^8 \cdot 3^2 \cdot 7 \cdot 17 + 1$	$1071^8 + 1$
F_6	$2^8 \cdot 5 \cdot 52562829149 + 1$	$262814145745^8 + 1$
F_9	$2^{16} \cdot 37 + 1$	$37^{32} + 1$
F_{11}	$2^{13} \cdot 3 \cdot 13 + 1$	$1248^{256} + 1$
F_{11}	$2^{13} \cdot 7 \cdot 17 + 1$	$3808^{256} + 1$
F_{12}	$2^{14} \cdot 7 + 1$	$448^{512} + 1$
F_{12}	$2^{16} \cdot 397 + 1$	$397^{256} + 1$
F_{12}	$2^{16} \cdot 7 \cdot 139 + 1$	$973^{256} + 1$
F_{18}	$2^{20} \cdot 13 + 1$	$208^{2^{14}} + 1$
F_{23}	$2^{25} \cdot 5 + 1$	$2560^{2^{19}} + 1$
F_{36}	$2^{39} \cdot 5 + 1$	$640^{2^{31}} + 1$
F_{38}	$2^{41} \cdot 3 + 1$	$1536^{2^{33}} + 1$
F_{73}	$2^{75} \cdot 5 + 1$	$10240^{2^{67}} + 1$

In addition to the information furnished by these fifteen Fermat prime factors it is known that F_0, F_1, F_2, F_3, F_4 are prime and that F_7 and F_8 are composite, as we have said above.

17. In a similar way (3), (4) and (10) yield factors of other simple binomial expressions. We shall merely illustrate this by means of two examples involving (3). Taking the first factor $2^7 \cdot 5 + 1$ of F_5 we have $2 \cdot 32 - 9 \cdot 7 = 1$. Hence $s = 9$, $r = 2$. Therefore

$$5^9 + 2 \equiv 0 \bmod 2^7 \cdot 5 + 1.$$

Again, taking $n = 9$ we may write the given factor in the form $2^{15} \cdot 74 + 1$. Then we have $8 \cdot 512 - 273 \cdot 15 = 1$; whence

$$74^{273} + 2 \equiv 0 \bmod 2^{16} \cdot 37 + 1.$$

SOME GEOMETRICAL RELATIONS OF THE PLANE, SPHERE, AND TETRAHEDRON.

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Professor Chrystal has remarked that mathematical works should be read backwards as well as forwards, *i. e.*, advanced notions of mathematics often throw light on elementary problems. While this "back tracking" is frequently brought out in analysis, it is seldom emphasized in connection with plane and solid geometry.

Many theorems of plane geometry have solid analogues which are but seldom developed. Especially is this true in the case of constructions which are slighted in solid geometry, owing to their theoretic nature. (The only way of actually drawing them being by the very indirect methods of descriptive geometry.)

For example, the problem of the "tangencies"—the construction of a *circle* in a plane from *three* elements, these being *points*, tangent *lines* or tangent *circles* —with its ten cases easily leads to the determination of a *sphere* from *four* elements, these being *points*, tangent *planes* or tangent *spheres*. There are fifteen cases in all, which are reduced to six by the following two principles, analogous to those used for the plane problem.

PRINCIPLE I.—If the given elements are only spheres and planes, and if the radii of the spheres be changed by that of one of the spheres, and if the planes be translated normal to themselves by a like amount, obviously the sphere determined by the new elements will be concentric with that determined by the original set. This principle serves to replace one of the tangent spheres by a point.

PRINCIPLE II.—If a sphere is tangent to two other spheres (sphere and plane) and goes through a given point, it goes through the inverse of this point, the center of inversion is a center of similitude of the two spheres (sphere and plane) and the radius of inversion is such as to make one of the spheres the inverse of the other (or of the plane). This also replaces one of the given spheres by a point.

The constructions follow; the proofs will be easily supplied by the reader familiar with the plane problem.

1. *Point, Point, Point, Point.*—The center of the required sphere is the intersection of the perpendicular bisecting planes of the joins of the points two and two. (Circum-sphere of a tetrahedron.) One solution.

2. *Plane, Plane, Plane, Plane.*—The center of the required sphere is the intersection of the planes bisecting the dihedral angles formed by the given planes. (In-sphere of a tetrahedron.) One solution.

3. *Point, Point, Point, Plane.*—Construct the perpendicular bisecting planes of the joins of the points intersecting in a line. Draw a sphere with center on this line to touch the given plane, and with the intersection of this line and the given plane as a "center" multiply this sphere to pass through the given points. Two solutions.

4. *Plane, Plane, Plane, Point.*—Draw the bisecting planes of the dihedral angles of the given planes, and draw a sphere with center any point on their line of intersection to touch the planes, then with the point of intersection of the given planes as a "center," multiply this sphere to pass through the given point. Two solutions.

5. *Point, Point, Point, Sphere.*—Draw any sphere through the three points to intersect the given sphere in a circle, and from the line of intersection of the plane of this circle with the plane of the three points draw a tangent plane to the given sphere, locating the point of contact of the required sphere. Then use (1). Two solutions.

6. *Plane, Plane, Plane, Sphere.*—Reduce by Principle I to (4). Four solutions.

7. *Plane, Plane, Point, Point.*—Locate the locus of the center as the intersection of the bisecting plane of the given dihedral angle and the perpendicular bisector of the join of the two points, draw a sphere with center on this line to touch the given planes, and with the intersection of the line of centers with the given planes as "center" multiply it to pass through the given points. Two solutions.

8. *Point, Point, Plane, Sphere.*—Reduce by Principle II to (3). Four solutions.

9. *Plane, Plane, Point, Sphere.*—Reduce by Principle II to (7). Four solutions.

10. *Point, Point, Sphere, Sphere.*—Reduce by Principle II to (5). Four solutions.

11. *Plane, Plane, Sphere, Sphere.*—Reduce by Principle I to (9). Eight solutions.

12. *Plane, Point, Sphere, Sphere.*—Reduce by Principle II to (3). Eight solutions.

13. *Point, Sphere, Sphere, Sphere.*—Reduce by Principle II to (5). Eight solutions.

14. *Plane, Sphere, Sphere, Sphere.*—Reduce by Principle I to (12). Eight solutions.

15. *Sphere, Sphere, Sphere, Sphere.*—Reduce by Principle I to (13). Sixteen solutions.

This last case can also be solved by a direct construction after Gergonne's construction for a circle tangent to three circles,[1] and, like it, in degenerate forms includes many remaining cases of the problem.

With center the intersection of the four radical planes of the spheres, and a radius equal to the tangent from this point to one of the given spheres, describe the sphere orthogonal to the given four spheres. Draw the planes of intersection of this sphere and each of the given spheres intersecting an "axial plane of similitude" of the spheres in four lines. Tangent planes from these lines to the corresponding spheres determine eight points of contact. The two spheres through these four and four are tangent to the given spheres. The eight "axial planes of similitude" determine the sixteen solutions.

The properties of plane triangles are often generalized for spherical triangles. These results, however, are unlike the ordinary ones of geometry, since they often involve trigonometric functions, replace straight lines by great circles, and are really a representation in Euclidean three-dimensional space of a non-euclidean plane.

The tetrahedron is the more natural spacial analogue of the plane triangle, but its properties are seldom brought out. The necessary and sufficient condi-

[1] "Recherche du cercle qui en touche trois autres dans un plan," *Annales de Mathématiques* vol. 7, 1817.—Editor.

tions for the co-planarity of six points on the different edges or for the concurrency of six planes through the different edges is easily obtained by applying Ceva's or Menelaus' theorem to three different faces, the condition being three equations in each case. The generalized "Ceva's theorem" might be used to prove the existence of the "axial planes of similitude" of four spheres mentioned above.

The existence of a median point (center of mean position or center of mass of the four vertices), *i. e.*, the intersection of the lines joining the centroids of the faces with the opposite vertices, and of the in- and circum-centers is usually proved, but the analogy is carried no further.

If an extension of an ortho-center be attempted, whether we define an altitude as a perpendicular from a vertex to the opposite face or as the join of a vertex with the ortho-center of the opposite face, we find that in general such altitudes do not intersect. The necessary and sufficient condition for either of these sets of "altitudes" being concurrent is the coincidence of the foot of one perpendicular with the ortho-center of the face to which it is perpendicular, in which case both sets of altitudes coincide, and the tetrahedron has its opposite edges perpendicular. Such a tetrahedron has many interesting properties.

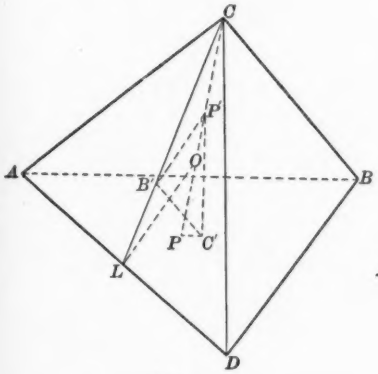


FIG. 1.

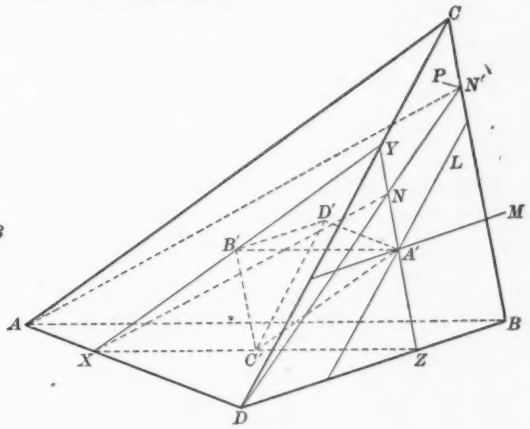


FIG. 2.

For example (Fig. 1). Figs. 1 and 2 are orthogonal projections of the spacial configurations they represent), if $ABCD$ is such a tetrahedron, the four centroids of the faces (as B', C'), the four feet of the altitudes (as P), and the four points dividing the segments of altitudes beyond the ortho-center in the ratio of $1 : 2$ (as P', such that $CP' = 2P'O$) all lie on a sphere, which might be called the "twelve-point" sphere after the "nine-point" circle.

Proof. $AO \perp$ Plane DBC

$DO \perp$ Plane ABC

$\therefore$ Plane $AOD \perp$ Planes DBC and ABC or to CB their intersection

$\therefore LO \perp CB$

But $CB' : B'L = CP' : P'O = BC' : C'L = 2 : 1$.

$\therefore LO \parallel B'P'$ and $CB \parallel B'C'$.

$\therefore B'P' \perp B'C'$.

But $PP' \perp PC'$;

$\therefore$ Sphere on diameter $C'P$ passes through B' (and similarly through remaining centroids of faces) and also through P and P'. By a similar proof, the sphere through the centroids may be shown to pass through the other points corresponding to P and P'.

Such a tetrahedron also possesses an "Euler" line. Since $A'B'C'D'$ is similarly situated with regard to $ABCD$ with respect to the median point M, T the

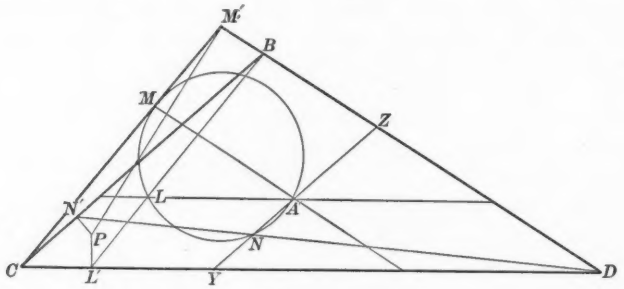


FIG. 3.

center of the twelve-point sphere, K the circum-center and M are collinear, and $KM : MT = 3 : 1$. Also since $OP' : OC = 1 : 3$, etc., the twelve-point sphere and the circum-sphere are similarly situated with regard to 0, and therefore O, T and K are collinear and $OT : OK = 1 : 3$. Also from the similar situation of $A'B'C'D'$ and $ABCD$ O, M and Q the intersection of the perpendiculars at the centroids of the faces are collinear and $MQ : MO = 1 : 3$. Therefore

O = the ortho-center,
T = the center of the twelve-point sphere,
M = the median point,
Q = the intersection of the perpendiculars at the centroids of the faces and
K = the center of the circum-sphere

are on line in the above order, and the segments $OT : TM : MQ : QK = 2 : 1 : 1 : 2$.

Conversely, solid constructions may give rise to plane theorems. If the centroids of the faces of any tetrahedron $ABCD$ (Fig. 2) $A'B'C'D'$ be located and a sphere passed through them, this sphere cuts plane $A'B'C'$ in the nine-point circle of triangle XYZ and therefore passes through N the foot of the altitude of this triangle on YZ. Similar points may be located corresponding to faces $A'B'D'$ and $A'C'D'$. But DN cuts BC in N', the foot of the A-altitude of triangle ABC,

and N' is also the foot of a perpendicular on BC from P, the foot of a perpendicular from A to BDC.

Since for different tetrahedra P becomes any point in plane BCD, the following theorem is obtained (Fig. 3): If from a point (P) perpendiculars be drawn to the sides of a triangle (BCD) produced if necessary, and their feet ($L'M'N'$) be joined to the opposite vertices, the three intersections of these lines with the parallels to the sides through the median point (LMN) and the median point (A') lie on a circle whose center divides the join of the circum-center of the original triangle and the center of mean position of the three vertices of the triangle and the arbitrary point externally in the ratio 1 : 4. The second part of the theorem follows from the fact that $B'C'D'$ and BCD (Fig. 2) are similarly situated with respect to the center of mass of the four vertices of the tetrahedron, and the orthogonal projection of the circum-center of $A'B'C'D'$ on plane BCD coincides with that of the circum-center of triangle $B'C'D'$.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTIONS.

37. Criticise the following as fundamental definitions of elementary geometry:

A plane surface is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A straight line is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

38. As the several courses in secondary and collegiate mathematics are now taught there is a noticeable difference of treatment with respect to the relative emphasis placed on logical accuracy and on development of technique. In elementary algebra, technique predominates, while in plane and solid geometry and advanced algebra logic is more strongly stressed. Trigonometry, analytic geometry, and the calculus are less easily classified; but there is at least a general tendency to emphasize logic in analytic geometry and technique in the calculus. It is suggested that a general appraisal of the reasons for this difference in treatment, its value, and possible alterations, would be helpful.

DISCUSSIONS.

THE EXISTENCE OF CUSPS ON THE EVOLUTE AT POINTS OF MAXIMUM AND MINIMUM CURVATURE ON THE BASE CURVE.

By G. H. LIGHT, University of Colorado.

The purpose of this paper is to show that whenever a curve $F(x, y) = 0$ has a point of maximum or minimum curvature the evolute has at the correspond-

ing point a cusp at which the curvature is infinite.[1] The assumption is made that at the point on the curve T under consideration $F_y \neq 0$.

Hence the value of dy/dx may be written

$$dy/dx = -F_x/F_y,$$

from which one easily obtains

$$d^2y/dx^2 = -B/F_y^3,$$

where

$$B = F_x^2F_{yy} - 2F_xF_yF_{xy} + F_y^2F_{xx}.$$

By definition, the curvature has the following value;

$$k = -\frac{B}{(F_x^2 + F_y^2)^{3/2}}.$$

Hence at a critical value its derivative must vanish; *i. e.*, $dk/dx = 0$. Performing the necessary work, it is found that

$$\frac{dk}{dx} = -\frac{(C - 3D)}{F_y(F_x^2 + F_y^2)^{5/2}},$$

where C and D have the values:

$$C = (F_x^2 + F_y^2)(F_y^3F_{xxx} - 3F_y^2F_xF_{xxy} + 3F_yF_x^2F_{xyy} - F_x^3F_{yyy}),$$

$$D = (F_xF_yF_{xx} + F_y^2F_{xy} - F_x^2F_{xy} - F_xF_yF_{yy})(F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy})$$

$$= F_y^4F_{xx}F_{xy} - 2F_y^3F_xF_{xy}^2 - F_y^3F_xF_{xx}F_{yy} + F_y^3F_xF_{xx}^2 - 3F_y^2F_x^2F_{xx}F_{xy}$$

$$+ 3F_y^2F_x^2F_{xy}F_{yy} + F_yF_x^3F_{xx}F_{yy} + 2F_yF_x^3F_{xy}^2 - F_yF_x^3F_{yy}^2 - F_x^4F_{xy}F_{yy}.$$

From this results

THEOREM 1. At a point of either maximum or minimum curvature the relation $C = 3D$ must hold.

Having found the condition which holds at maximum and minimum points, the nature of the corresponding point on the evolute will be investigated. The equation of the evolute will be written in parametric form, x being regarded as the parameter:

$$\alpha = x - \frac{F_xF_y^2 + F_x^3}{B}, \qquad \beta = y - \frac{F_x^2F_y + F_y^3}{B}.$$

The curvature, K_1, of the evolute is given by the formula

$$K_1 = \frac{d^2\beta}{d\alpha^2}\Big/\left[1 + \left(\frac{d\beta}{d\alpha}\right)^2\right]^{3/2}$$

and is readily computed when it is remembered that

$$d\beta/d\alpha = F_y/F_x$$

[1] Cf. Goursat-Hedrick, *A Course in Mathematical Analysis*, vol. I, p. 438.

and therefore

$$\frac{d^2\beta}{d\alpha^2} = \frac{d\left(\frac{d\beta}{d\alpha}\right)}{dx} \cdot \frac{1}{\frac{d\alpha}{dx}}.$$

After some computation, it is found that

$$\frac{d\alpha}{dx} = \frac{F_x(C - 3D)}{F_y \cdot B^2}. \tag{1}$$

Likewise

$$\frac{d\beta}{dx} = \frac{C - 3D}{B^2} \tag{2}$$

and

$$\frac{d^2\beta}{d\alpha^2} = \frac{B^3}{F_x^3} \cdot \frac{1}{C - 3D}.$$

Therefore

$$K_1 = \frac{B^3}{(F_x^2 + F_y^2)^{3/2}} \cdot \frac{1}{C - 3D}. \tag{3}$$

From (1), (2), (3) it is seen that when $d\alpha/dx$ and $d\beta/dx$ vanish, K_1 becomes infinite and a cusp occurs on the evolute.

THEOREM 2. When the curvature of T is either a maximum or a minimum (at a point where neither F_x nor F_y is infinite) the relation $C = 3D$ must be satisfied. Moreover, the evolute of the normals has a cusp at the same time and its curvature is infinite. Furthermore, since at a point of maximum curvature the radius of curvature is a minimum, it is seen that the corresponding point on the evolute is nearer T than at any other time and that the cusp points *towards* T. In like manner, at a point of minimum curvature the cusp points *away* from T.

As a simple example consider the ellipse.

Here

$$F(x, y) \equiv b^2x^2 + a^2y^2 - a^2b^2 = 0$$

and

$$F_x = 2b^2x, \qquad F_y = 2a^2y, \qquad F_{xy} = 0,$$
$$F_{xx} = 2b^2, \qquad F_{yy} = 2a^2, \qquad F_{xxy} = 0,$$
$$F_{xxx} = 0, \qquad F_{yyy} = 0, \qquad F_{xyy} = 0,$$

from which

$$C = 0, \qquad D = -64a^6b^6c^2xy$$

and

$$C - 3D = 192a^6b^6c^2xy.$$

This vanishes only when $x = 0$ or $y = 0$, and verifies the fact that maximum and minimum curvature occurs only at the vertices of the ellipse.

From the well-known form of the evolute of the ellipse,

$$(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3},$$

the curvature, K_1, is found to be

$$K_1 = -\frac{a^{2/3}b^{2/3}(\alpha - \beta)}{3\beta^{2/3}\alpha^{2/3}(b^{4/3}\alpha^{2/3} + a^{4/3}\beta^{2/3})^{3/2}},$$

and this becomes infinite only when $\alpha = 0$ or $\beta = 0$. Thus cusps occur on the evolute at the points $\left(\pm\frac{c^2}{a}, 0\right)$, $\left(0, \pm\frac{c^2}{b}\right)$ corresponding to points of maximum and minimum curvature on the ellipse.

RECENT PUBLICATIONS.

REVIEWS.

Exercises in Algebra (including trigonometry). By T. P. Nunn. 2 volumes. London and New York, Longmans, 1913–1914. Vol. I, 11 + 421 pp. Vol. II, 11 + 551 pp. Price $4 + 6\frac{1}{2}$ shillings.

The Teaching of Algebra (including trigonometry). By T. P. Nunn. London and New York, Longmans, 1914. 12mo, 14 + 616 pp. Price $7\frac{1}{2}$ shillings.

Among the contributions that have recently been made to the solution of the problem of teaching secondary and collegiate mathematics, Dr. Nunn's work is one of the most significant. His plan is to build up a course which shall combine the most vital and essential parts of algebra, trigonometry, analytic geometry, and elementary calculus, and present them in a way which shall meet the needs both of the student who is studying mathematics merely for its cultural, disciplinary, or informational value, and of the student who plans to go on to further study of the subject. Many teachers have serious doubts as to the wisdom of giving the same work to these two different classes of students; but Dr. Nunn's course seems to the reviewer to furnish a successful demonstration that the thing can be done.

The two volumes of *Exercises* are designed for the student, and the third volume, on *Teaching*, indicates to the teacher the reasons for the order in which topics are taken up, and gives suggestions as to methods of presentation. The first volume of the *Exercises* contains little detailed explanation for the student, this being left for the teacher to give verbally on lines indicated clearly in the *Teaching*. The second volume of the *Exercises*, however, contains more discussion of the topics treated, with the object of developing in the student the ability to read and understand mathematical writing.

The work in elementary algebra is centered about two topics, the graph and the formula. By making skillful and persistent use of both, the author gives a vitality and meaning to the fundamental laws of operation which the student can rarely gain through the mere study of the axioms and formal laws. To give one illustration, the distributive law of multiplication is not first *stated*, and then

illustrated by geometrical figures, as is done in most texts, but rather it is *developed* in the mind of the student as a convenient short cut in area problems that require the value of $ac \pm bc$. Thus the usual order of procedure is here reversed, factoring being considered *before* multiplication as a formal algebraic process. Furthermore, the formal laws are first used with signless numbers, and not until the second section of the course (27th exercise, page 155) are directed numbers introduced. Much is made of approximation formulas as illustrations of the usefulness of such identities as $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

Direct proportion is introduced early, as a particularly important relation between variables, and the opportunity is taken to define the trigonometric functions of acute angles. This is surely an admirable idea, and offers a suggestion that might well be considered by American writers of texts on elementary algebra. After the introduction of directed numbers, arithmetic series are studied, and the summation applied to the problem of finding certain areas in a manner that prepares the ground very skillfully for the elementary work in integral calculus which is to follow not much later. After an elementary study of the quadratic equation and the principles of transformation of coördinates by translation, the remarkable innovation is introduced of taking up "area functions" (Exercise 47, p. 250). The method employed is the development of what the author calls "Wallis's Law," and may be illustrated as follows: in order to obtain the area under the curve $y = kx^2$, from $x = 0$ to $x = x$, the student is led to discover by induction that the ratio of the required area to that of the rectangle of base x and height kx^2 is approximately

$$\frac{0^2 + 1^2 + 2^2 + \cdots + m^2}{(m+1)m^2} = \frac{2m+1}{6m} = \frac{1}{3} + \frac{1}{6m},$$

where m is the number of subdivisions of the base. The conclusion is, that we may with no appreciable error take $\frac{1}{3}kx^3$ as the area under the curve. Similar work is carried through for other areas. At about this same period the subject of gradients and simple differentiation formulas are also introduced. It certainly demands great skill on the part of the teacher to present this work to boys of 15 and 16 years, and it would be interesting to know if the experiment has been tried in any American secondary schools.

The subject of logarithms is introduced by means of "growth problems," that is, problems depending upon the value of a^x for certain integral or fractional values of x. The historical method of Napier in constructing his system of logarithms is then employed, a procedure that seems rather questionable, but which is certainly worthy of careful experimental comparison with the usual method of approach.

Part II of the *Exercises* carries the student much farther into the field of what we know as collegiate mathematics, including functions of two variables, the trigonometry of the sphere, complex numbers, periodic functions (this section containing also a very good geometric introduction to the hyperbolic functions), and limits, which includes linear differential equations, Taylor's Theorem, and

even a good presentation of Weierstrass's derivativeless continuous function, of Peano's space-filling curve, and of Moore's "crinkly curve."

Constant use is made of the applications of mathematics. Thus, in Part I the introductory topics, the graph and the formula, are employed in connection with problems from mensuration, commercial arithmetic, physics, engineering, botany, and astronomy. Navigation is begun as a simple illustration of the use of the trigonometric functions (problems in parallel sailing, pages 118, 119) and is continued and elaborated in Part II, pages 105–142. In the latter place we find Sanson's, Lambert's, and Mercator's projections explained and applied in numerous problems; further, great circle sailing, the construction and use of gnomonic projections, and the elementary principles of practical astronomy (pages 143–169) are considered. The increasing importance of navigation in our national life suggests the advisability of enriching our mathematics courses, both in secondary school and in college, with much of this material. The subject of annuities and life insurance occupies pages 63–78 of Part II, compound interest having been considered briefly in Part I as an application of geometric progressions. The usually under-valued subject of statistics is given a very satisfactory treatment at the end of Part II (pages 431–514).

The mere mention of these topics is sufficient to show that the author has laid out an ambitious program for the secondary school course; but the original and stimulating manner in which all of the topics are presented tends to remove one's doubts as to the feasibility of carrying the young student into so much of the work which we have usually placed much later in the mathematics course. At any rate, we have in Dr. Nunn's work a valuable contribution to a most important problem, and one that can not fail to be suggestive and stimulating to any teacher of mathematics. The three volumes should find a place in every high school and college library.

R. B. McClenon.

Interpolation Tables, or Multiplication Tables of Decimal Fractions, giving the product to the nearest unit of all numbers from 1 to 100 by 0.01 to 0.99 and from 1 to 1,000 by 0.001 to 0.999. By Henry B. Hedrick. Carnegie Institution of Washington. Washington, 1918. 9 + 139 pp. Folio. Price $5.00.

Until about the beginning of the eighteenth century the multiplication of one number by another must have been considered a serious undertaking. In the old arithmetics very simple examples of the processes are given with as painful detail as is now given to an example in the extraction of the cube root. Napier's "bones" were hailed as a wonderful invention and yet they are of little use to one who knows his multiplication table as far as the nines. As late as 1841 an immense and complicated table was published by Bretschneider giving the product of all numbers up to 100,000 by 2, 3, 4, 5, 6, 7, 8 and 9. Nevertheless, as early as 1610 a table giving the product of any two numbers less than 1,000 was published by Herwart von Hohenburg in an immense folio volume of over a thousand pages.

In 1864 Crelle published a table which is still an indispensable part of every computer's library.[1] It gives in about five hundred pages the product of any two numbers less than 1,000 and may easily be used to multiply any two numbers together. It is also conveniently used as a division table.

The tables prepared by Dr. H. B. Hedrick and published by the Carnegie Institution of Washington are intended to cover the same range of numbers as Crelle's tables using 139 pages instead of 500. The abridgment is obtained by giving only three significant figures in the product, which for many purposes is sufficient. The arrangement of the table is also new in that while one factor is listed at the top of the page the other is found in the body of the table, the product being listed at the side of the page. Only a few numbers are necessary to list in the body of the table, since if only the first three significant figures are desired the products of .293, for example, by 279, 280 and 281 are identical. Only the smallest, 279, is listed. This effects a considerable saving of space.

The reviewer is not convinced that the smallness of the table is a sufficient advantage over Crelle's table to offset the manifest disadvantages. In order to test the matter he made a list of one hundred products of three digits by three digits. He first found the products "by hand" using his own method of cross-multiplication by which the product is obtained from the left. (See *Science*, July 11, 1902.) The products were then found from Dr. Hedrick's table, then from Crelle's table, and finally by means of the Monroe Calculating Machine. The results were as follows:

By hand	28 minutes.
By Hedrick's table	32 minutes.
By Crelle's table	29 minutes.
By Monroe machine	26 minutes.

The reviewer has not had any practice with Crelle's table for many years, and was so unused to it that he had to read the directions for using it. He has had, quite recently, considerable practice in using the calculating machine, so that probably he would be a little faster than the average engineer in using it. The comparison with Crelle's table is interesting as bearing out what the reviewer has always maintained, that is, that if one has to turn over pages it is as easy and convenient to turn over five hundred pages as one hundred, and that there is no appreciable saving in time in using a four place table of logarithms instead of a six place table, and when interpolations are necessary there is a distinct loss of time and accuracy in using the smaller table.

D. N. LEHMER.

A Treatise on Differential Equations. By A. R. FORSYTH. Fourth edition. London, Macmillan, 1914. 8vo, 584 pp. Price 17 shillings.

Solutions of the Examples in a Treatise on Differential Equations. By A. R. FORSYTH. London, Macmillan, 1918. 8vo, 249 pp. Price 10 shillings.

Early in the eighties Macmillan and Company consulted Forsyth with regard to the preparation of a new edition of Boole's *Treatise on Differential Equations*, a work which had already

[1] Other multiplication tables are described on page 327 of J. W. L. Glaisher's article, "Table, Mathematical," in the eleventh edition of the *Encyclopædia Britannica*—EDITOR.

gone through four editions. The result was that Forsyth gave them, in 1885, a work of his own with the same title. The second edition appeared in 1888; the third, much enlarged, in 1903, and the last, eleven years later, still further increased in size. The Italian translation, by Arbicone, of the second English edition was published in 1901[1] and the German edition by Maser (with an appendix containing solutions of the problems) in 1889; the second German edition came out in 1912,[2] but the solutions given were only of the problems of the second English edition.

In preparing his volume of solutions Forsyth worked out each example himself and found that, "with possibly three exceptions, all the equations have proved soluble." He remarks further: "The first German edition . . . translated by the late Dr. H. Maser contained the solutions of most of the examples in my own first and second editions—not a few of them were supplied to him by me. But my later editions contained a vast added array of equations, not solved in the second German edition. To make the volume complete, all the questions (except where the results were taken from original memoirs) were worked out anew from the beginning."

"I have had in mind principally the needs of teachers and only subsidiarily the case of students who are in a position to follow courses of instruction."

Text-book on Navigation and Nautical Astronomy. By J. GILL. Revised and enlarged by W. V. Merrifield. New edition. London, Longmans, 1918. 8vo, 8 + 438 + 12 pp. Price 20 shillings.

Contents: Part I, pp. 1–262: Projections; definitions and instruments; logarithms; solution of plane triangles; solution of spherical triangles; the sailings; tides; reduction of soundings; time; elements from the 'Nautical Almanac'; sidereal and solar time; corrections of altitudes; latitude by meridian altitudes; latitude by ex-meridian altitude; compass error by amplitudes; compass error by azimuths; longitude by chronometer; Sumner's method of projection; the chart; use of Napier's diagram; great circle sailing; latitude by double altitudes; finding error of chronometer; calculation of altitudes; longitude by linear distances. Part II, pp. 263–342: Construction of charts; laws of storms; magnetism and deviation of the compass; syllabus; miscellaneous problems. Appendix, pp. 343–402. Answers, 403–438. Tide Tables 1–12.

Quotations from the preface to the new edition: "The new features in this edition are—(1) Rearrangement, by which the proofs of the various problems are inserted at the beginning of the separate chapters instead of being collected in one chapter at the end of the book. . . . (2) All the astronomical examples are fresh, the necessary rules are given for working them, and, where possible, an example is worked direct from the figure, without using special formulæ. . . . (3) The necessary elements from the 'Nautical Almanac' are embodied in the book in the form found in the 'N. A.' . . . (4) The chapters on trigonometry have been entirely rewritten. . . . (5) New chapters have been introduced on the projections by which any figure may be drawn to scale, and on the correction of altitudes. (6) The chapters on tides and tidal soundings have been rewritten to suit the recent tide tables, extracts from which are found at the end of the book."

A Vest-pocket Handbook of Mathematics for Engineers. By L. A. WATERBURY, with special sections by G. A. GOODENOUGH and H. H. HIGBIE. Third edition, enlarged (total iss e twelve thousand). New York, Wiley, 1918. 16mo, 13 + 278 pp. Flexible cover, gilt. Price $1.50.

The first edition of this *Handbook* was in 1908; tables were added in 1909. To the second edition in 1915 sections on hydraulics and reinforced concrete, and a table of conversion factors were added. The principal additions to the present edition are the sections on heat engineering by Professor Goodenough, and on electrical engineering by Professor Higbie, with the corresponding related tables. The contents of the work are as follows: Algebra (exponents and logarithms; proportion; progressions; quadratic, cubic, higher equations; graphical solutions of

[1] *Trattato sulle equazioni differenziale.* Prima versione dall'inglese di Alfredo Arbicone. Livorno, 1901. 12 + 337 pp.

[2] *Lehrbuch der Differentialgleichungen.* Von A. R. FORSYTH (mit den Auflösungen der Aufgaben von Hermann Maser). Zweite Auflage, nach der dritten des englischen Originals besorgt und mit einem Anhang von Zusätzen versehen von W. Jacobsthal. Braunschweig, Vieweg, 1912. 22 + 921 pp. Reviewed by R. D. CARMICHAEL in *Bull. Amer. Math. Soc.*, Vol. 19, pp. 256–259. Feb., 1913.

equations), pp. 1–7; Trigonometry (plane and spherical triangles; hyperbolic functions), 8–13; Analytic Geometry (transformation of coördinates; straight line; circle; parabola; ellipse; hyperbola; cycloid; miscellaneous curves; solids), 14–22; Differential and integral calculus, 23–38; Theoretical mechanics, 39–53; Mechanics of materials, 54–91; Hydraulics, 92–101; Heat engineering, 102–123; Electrical engineering formulæ, 124–155. Tables (logarithms of numbers, sines, cosines, tangents and cotangents; natural sines, cosines, tangents and cotangents; conversion factors; properties of saturated steam (Goodenough); pressure-entropy table for steam (Goodenough), 156–269.

La Vie universitaire à Paris. Ouvrage publiè sous les auspices du Conseil de l'Université de Paris. Avec 92 photographies hors texte. Paris, Colin, 1918. Royal 8vo, 231 pp. Price 12 francs.

The two chapters of this sumptuous work which are of especial interest to the mathematician are chapter 4: "La faculté des sciences" by Maurice Caullery, 50–70; and chapter 5: "L'Ecole Normale Supérieure" by E. Lavisse, 71–74. Plate 15 presents Pirou's well-known portrait of Hermite and a much less familiar portrait of Poincaré by Manuel. There are also interesting views of the Sorbonne.

Business arithmetic. By C. W. SUTTON and N. J. LENNES. Boston, Allyn and Bacon, 1918. 8vo, 6 + 466 + 8 pp. Price $1.40.

Brief business arithmetic. By C. W. SUTTON and N. J. LENNES. Boston, Allyn and Bacon, 1918. 8vo, 5 + 297 + 6 pp. Price $1.16.

The titles of the last twenty-one chapters of the larger work are as follows: Promissory notes, partial payments, banks and bank discount, cash balance in bank, savings accounts, stocks and shares, bonds, the stock exchange, graphic representation, domestic exchange, foreign exchange, equation of payments and accounts, property insurance, life insurance, direct taxation, income tax, United States customs, partnership, storage, building and loan associations, and depreciation.

NOTES.

The United States Ordnance Department is to publish a treatise on exterior ballistics. It is being prepared for the press under the direction of Captain Dunham Jackson.—Teubner announces the publication of: C. CRANZ, *Lehrbuch der Ballistik.* Herausgegeben unter Mitwirkung von K. Becker. Band 4: Sammlung von Zahlentafeln, Diagrammen, und Lichtbildern. 2. vermehrte Auflage, Leipzig, 1918.

It is announced that the fourth, fifth, and sixth volumes of the *Handbuch der angewandten Mathematik* edited by H. E. Timerding (Leipzig, Teubner) are to be respectively entitled: *Die graphischen Methoden der technischen Mechanik, Mathematische Statistik und Versicherungsrechnung,* and *Grundzüge der Astronomie.* The first three volumes, published in 1914–15, were *Praktische Analysis* by H. von SANDEN, *Darstellende Geometrie* by J. HJELMSLEV, and *Grundzüge der Geodäsie* by M. NÄBAUER.

Three bulletins have been issued by the Naval Consulting Board of the United States and the Engineering Council's War Committee of Technical Societies. In *Problems for Aëroplane Improvement,* bulletin No. 3, August, 1918, there is a "Working Bibliography," pages 19–29. This bibliography includes portions of one published by *Aviation and Aëronautical Engineering,* May 1, 1917, entitled "Two hundred books on aëronautics" by E. N. Fales. Paul Brockett's com-

prehensive volume, *Bibliography of Aëronautics* published by the Smithsonian Institution in 1910, is now being brought up to date.

Macmillan's spring announcements include an entirely new edition of CAJORI'S *History of Mathematics;* a new work in the Hedrick series, KENYON and INGOLD'S *Elemements of Plane Trigonometry; A short Course in Mathematics for Freshmen* by R. E. MORITZ; and an elementary work by BERTRAND RUSSELL entitled *Introduction to Mathematical Philosophy* (published by Allen and Unwin, London)—The Cambridge University Press announces that J. H. JEANS'S *Problems of Cosmogony and Stellar Dynamics* is soon to appear.

Teubner announces the completion of F. Dingeldey's edition of the genial Fiedler's free translation of Salmon's *Conic Sections*. The first English edition was published in 1848, and the first in German in 1860. This translation has appeared in two parts since the fifth edition in 1887. Dingeldey's edition constitutes the eighth edition (1915) of the first part, and the seventh edition (1918) of the second part—a work of over 900 pages. The references to sources are very numerous.

Salmon was born at Dublin in 1819 and graduated from Trinity College there in 1841 when he immediately became a fellow and lecturer in mathematics. He was made a divinity lecturer as well in 1845. His first mathematical paper was in 1844 and his first publication in a theological subject in 1849. He was regius professor of divinity from 1866 till he became provost in 1888. He died in 1904. From Fiedler's "Zum Gedächtnis George Salmons" (pp. vi–xv of the first part) the following sentences are quoted: "He was a master in chess and was once chosen as an opponent of Morphy whom he met successfully. I have already mentioned that he was a great reader along many lines including much of a lighter character. . . . So well was he acquainted with the literature for young people, he was able to contribute to the 39th volume of the *Fortnightly Review* an interesting inquiry on the theme, 'What boys read'; he held the girls' book, *Alice in Wonderland*, in high esteem. Above all things he rejoiced in humor—both the very refined and the more rugged." In John Ossory's sketch in the *Dictionary of National Biography* occur the lines: "Hospitable and kindly, Salmon had many friends and interests. In youth a competent musician and a chess player of remarkable power, he cultivated both recreations until an advanced age. He was always an omnivorous reader (except in the two departments of metaphysics and poetry, for which he had no taste) and had a special affection for the older novelists, being accustomed to recommend the study of Jane Austen as a liberal education. The homely vigor and the delightful wit of the long letters which he was accustomed to write to his friends entitle him to rank as one of the best letter-writers of the last century."

Three parts of the *Encyklopädie der mathematischen Wissenschaften* were published in 1918: (1) Band III 1, Heft 6: M. Zacharias, "Elementargeometrie und elementare nichteuklidische Geometrie in synthetischer Behandlung." Zweiter Theil mit Zusätzen von W. F. Meyer; (2) Band III 2, Heft 7: C. Segre, "Mehrdimensionale Räume"; (3) Band VI 1 B, Heft 4: E. von Schweidler, "Atmosphärische Elektrizität"; A. Schmidt, "Erdmagnetismus und verwandte Erscheinungen." There are now eight complete volumes of the *Encyklopädie;* the two for "Arithmetik und Algebra," two of the four volumes for "Analysis," and four of the five volumes for "Mechanik" (the Registerband alone is lacking). At present an extra charge, amounting to 30 per cent. of old prices, is made for all parts of the *Encyklopädie*.

Teubner has recently published two additional parts of Materialien für eine

wissenschaftliche Biographie von Gauss—Heft 4: *C. F. Gauss als Zahlenrechner* von A. Galle (Pp. 1–24); Heft 5: *C. F. Gauss als Geometer* von P. Stäckel (Pp. 25–142), 1918. Earlier parts were—Heft 1: *Ueber Gauss' zahlentheoretische Arbeiten* von P. Bachmann, 1911; Heft 2: *C. F. Gauss, Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799,* and Heft 3: *Ueber Gauss' Arbeiten zur Funktionentheorie,* von L. Schlesinger, 1912.

The first part of the tenth volume of Gauss' collected works appeared in 1917. It is a volume of nearly 600 pages and is entitled "Nachträge zur reinen Mathematik. Nachbildung und Abdruck des Tagebuchs." The second part of volume 10 (Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der reinen Mathematik—Zahlentheorie, Funktionentheorie, Algebra, Geometrie, Variationsrechnung) and the first part of volume 11 (Nachträge zur Physik, Astronomie und Chronologie) are in the press. The second part of volume 11 which is in course of preparation is to contain "Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der angewandten Mathematik." The twelfth and last volume is to be biographical.

The following quotation is a bit of personal testimony to the value of mathematical study, from the pen of one who has become world renowned for the clarity and beauty of his literary style:

"Geometry was to teach me the logical progression of thought; it was to tell me how the difficulties are broken up into sections which, elucidated consecutively, together form a lever capable of moving the block that resists direct efforts; lastly, it showed me how order is engendered, order, the base of clarity. If it has ever fallen to my lot to write a page or two which the reader has run over without excessive fatigue, I owe it, in great part, to geometry, that wonderful teacher of the art of directing one's thought. True, it does not bestow imagination, a delicate flower blossoming none knows how and unable to thrive on every soil; but it arranges what is confused, thins out the dense, calms the tumultuous, filters the muddy, and gives lucidity, a product superior to all the tropes of rhetoric.

"Yes, as a toiler with the pen, I owe much to it. Wherefore my thoughts readily turn back to those bright hours of my novitiate, when, retiring to a corner of the garden in recreation-time, with a bit of paper on my knees and a stump of pencil in my fingers, I used to practise deducing this or that property correctly from an assemblage of straight lines. The others amused themselves all around me; I found my delight in the frustum of a pyramid." From *The Life of the Fly with which are interspersed some chapters of autobiography* by J. Henri Fabre, translated by A. T. de Mattos. New York, Dodd, Mead and Co., 1913, Chapter 12: "Mathematical memories: Newton's binomial theorem."

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 40, no. 4, October, 1918: "Theta modular groups determined by point sets" by A. B. Coble, 317–340; "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order—a particular solution" by W. V. N. Garretson, 341–350; "A collineation group isomorphic with the group of the double tangents of the plane quartic" by C. C. Bramble, 351–365; "Proof of Pohlke's theorem and its generalizations by affinity" by A. Emch, 366–374; "Arithmetical theory of certain Hurwitzian continued fractions" by D. N. Lehmer, 375–390.

Bulletin of the American Mathematical Society, volume 25, no. 4, January, 1919: "On the evaluation of the elliptic transcendents η_2 and η_2'" by Harris Hancock, 150–157; "On plane algebraic curves with a given system of foci" by Arnold Emch, 157–161; "Quadratic systems of circles in non-Euclidean geometry" by D. M. Y. Sommerville, 161–173; "Continuous sets that have no continuous sets of condensation" by R. L. Moore, 174–176; "Derivativeless continuous functions" by M. B. Porter, 176–180; Review by R. D. Carmichael of Picard's *Les*

sciences mathématiques en France depuis un demi-siècle (Paris, 1912), 180–184; Review by A. D. Pitcher of Young and Morgan's *Elementary Mathematical Analysis* (New York, 1917), 185–187 —No. 5, February: "The December meeting of the San Francisco section," B. A. Bernstein, 193–196; "The scientific work of Maxime Bôcher" by G. D. Birkhoff, 197–215 [contains a list of Bôcher's writings, 101 titles]; "On a theorem of oscillation" by W. F. Osgood, 216–221; "Proof of a property of the norm of a cyclotomic integer" by H. S Vandiver, 221–223; "Trajectories and flat points on ruled surfaces" by J. K. Whittemore, 223–229; "Points of view of Cauchy and Weierstrass in the theory of functions" by R. D. Carmichael, 230–234 [a review of E. Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe*]; Review by R. D. Carmichael of G. W. Parker's *Elements of Optics for the use of schools and colleges* (London, 1915), 234–235; "Notes" and "New Publications," 235–240—No. 6, March: "The twenty-fifth annual meeting of the American Mathematical Society," 241–257; "Applications of the theory of summability to developments in orthogonal functions" by C. N. Moore, 258–276; "Modular systems" by R. D. Carmichael, 276–279 [a review of F. S. Macaulay's *The algebraic theory of modular systems* (Cambridge, 1916)]; "Notes" and "New Publications," 279–288.

EDUCATION, Boston, volume 39, no. 4, December, 1918: "On the disciplinary and applied value of mathematical study" by C. N. Moore, 209–216.

EDUCATIONAL REVIEW, New York, volume 57, no. 2, February, 1919: Review by C. J. Keyser of Shaw's *Lectures on the Philosophy of Mathematics* (Chicago, 1918), 167–169.

JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER VEREINIGUNG, volume 27, nos. 1–4, January–April, 1918: "Analyse und Synthese in der Geometrie" by L. Heffter, 1–19 [Rede bei Uebernahme des Prorektorats der Universität Freiburg i. B. am 12. Mai, 1917]; "Verallgemeinerungen des Pohlkeschen Satzes" by E. Kruppa, 20–36; "Ueber algebraische Gleichungen mit lauter reellen Wurzeln" by J. v. S. Nagy, 37–43; "Ueber geometrische Relationen zwischen den Wurzeln einer algebraischen Gleichung und ihrer Derivierten" by J. v. S. Nagy, 44–48; "Ueber nichtlineare Integralgleichungen vom Volterraschen Typus" (Zusätze zu der Arbeit in Bd. 23, 1914, S. 303–313) by J. Horn, 48–53; "Ein funktionentheoretischer Satz Jacobis "by G. Kowalewski, 53–55; "Zu Grassmanns Note: 'Lösung der Gleichung $x^3 + y^3 + z^3 + u^3 = 0$ in ganzen Zahlen'" by A. Hurwitz, 55–56; "Ein grundlegender Satz von Poncelet über die Brennpunkte von Kegelschnitten und seine Ausdehnung auf kubische Raumkurven," 57–66; "Ueber Kurven gleichmässigster Krümmung" by R. v. Mises, 67–71; "Der mittlere Fehler des zum Quadrat erhobenen Divergenzkoeffizienten" by L. v. Bortkiewicz, 71–80.—Nos. 5–8. May–August, (publ. in October): "Der mittlere Fehler des zum Quadrat erhobenen Divergenzkoeffizienten" (conclusion) by L. v. Bortkiewicz, 81–126; "Neue Beiträge zur Charakterisierung der Riemannschen Transzendenten" by R. König, 126–141; "Grenzübergänge in der Krümmungslehre" by P. Stäckel, 142–157; "Mittelwertsätze der Potentialtheorie" by W. Blaschke, 157–160; "Bemerkung zu meinem Aufsatz über einen funktionentheoretischen Satz Jacobis" by G. Kowalewski, 160.

JOURNAL OF THE FRANKLIN INSTITUTE, volume 186, August and November, 1918: "A surface having only a single side" by C. Hering, 233–241, 627–630. *Also in Scientific American Supplement,* December 21, 1918. [Quotation: "It is quite simple, however, to devise a true surface which has only one side, and as this has some very odd and peculiar properties the following description of it may be of interest; and when this surface is better understood perhaps some applications of it may be found. Steinmetz in the *Journal of the American Chemical Society,* May, 1918, p. 733, for instance, showed how this and the Riemann surface, on account of their peculiar properties, can be used for illustrating some of the complicated relations in the periodic system of the chemical elements."]

MESSENGER OF MATHEMATICS, volume 48, no. 2, June 1918: "The problem of the square pyramid" (concluded) by G. N. Watson, 17–22; "Note on the representation of the expansion of a bordered determinant" by T. Muir, 23–32.

MONIST, volume 29, no. 1, January, 1919: "The relation of logic to mathematics" by R. A. Arms, 146–152; "Independence proofs and the theory of implication" by V. F. Lenzen, 152–160.

MONTHLY WEATHER REVIEW, Washington, D. C., volume 46, October, 1918: "A much needed change of emphasis in meteorological research" by W. S. Franklin, 449–453. [The subheadings of the article are: Errors of measurement—probable errors and probable departures; hydraulics as illustrating the method of mechanics; thermodynamics and the atomic theory; systematic physics and statistical physics; the postulate of indeterminism; the conception of a physical system as an atomic aggregate and reason for steadiness of behavior; meteorology.]

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 77, 1918, September: "Albert Gauthier-Villars," 321–322; "Sur le système de n équations du second degré

$$\begin{cases} x_1(x_2+x_3+\cdots+x_n)+1\cdot 2(x_1+x_2+\cdots+x_n)^2=3^2a^2, \\ x_2(x_1+x_3+\cdots+x_n)+2\cdot 3(x_1+x_2+\cdots+x_n)^2=5^2a^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n(x_1+x_2+\cdots+x_{n-1})+n(n+1)(x_1+x_2+\cdots+x_n)^2=(2n+1)^2a^2" \end{cases}$$

by R. Alezais, 323–342; "Sur une propriété caractéristique des coniques homofocales" by R. Bricard, 343–349; "Relations entre les rayons de courbure de deux courbes affines" by F. Balitrand, 349–352—October: "Sur la distribution des nombres premiers absolus" by E. Jablonski, 361–372; "Propriété des coniques et des quadriques à centres" by M. Weill, 373–383; "Sur les cercles de Pappus: formule de Pappus, formule de Schubert généralisée" by G. Fontené, 383–390; "Sur la condition pour que les tangents aux pieds des normales issues d'un point à une ellipse touchent un cercle" by F. Balitrand, 390–392; "Remarques géométriques sur la question de concours de l'Ecole Polytechnique en 1918" by P. du Plessis, 392–394.

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, series 2, volume 16, April, 1918: "Jean Gaston Darboux (1842–1917)" by A. R. F[orsyth], xliv–xlix; William Henry Besant (1828–1917)" by A. S. R[amsey], l–liii. To Besant are due the mathematical terms, glissette and orthocenter; "he endeavoured to substitute phoronomy for kinematics, but the latter was too firmly established to be changed."

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE UNITED STATES OF AMERICA, volume 4, no. 12, December, 1918: "On Jacobi's extension of the continued fraction algorithm" by D. N. Lehmer, 360–364; "A characterization of Jordan regions by properties having no reference to their boundaries" by R. L. Moore, 364–370.

PROCEEDINGS OF THE ROYAL SOCIETY, London, volume 94A, 1918, no. A662, June: "Jons Oskar Backlund (1846–1916) by H. H. T[urner], pp. xx–xxiv; no. A663, July: "J. G. Darboux (1842–1917)" by E. W. H[obson], pp. xxxi–xxxiv and portrait.

REVUE DE MÉTAPHYSIQUE ET DE MORALE, volume 25, no. 3, May–June, 1918: "La théorie de la relativité et le temps universel" by E. Guillaume, 285–323.

REVUE PHILOSOPHIQUE DE LA FRANCE ET DE L'ÉTRANGER, volume 43, 1918, November–December: "Reflexions sur la thermodynamique à propos d'un livre récent" [L. Selme's *Principe de Carnot contre formule empirique de Clausius, essai sur la thermodynamique* (Paris, 1917)] by L. Rougier, 435–478.

SCHOOL AND SOCIETY, volume 8, 1918, December 21: "The History of Science" by L. C. Karpinski, 741–749. [Review of Marvin's *The Living Past, A Sketch of Western Progress*; Libby's *An Introduction to the History of Science*; Sedgwick and Tyler's *A Short History of Science, Studies in the History and Method of Science* edited by C. Singer; Barnard's *The Casting-Counter and the Counting-board*; Peters and Knobel's *Ptolemy's Catalogue of Stars, A Revision of the Almagest*; Knobel's *Ulugh Beg's Catalogue of Stars*; and Cajori's *William Oughtred*]. "The Reed I. Q. Slide Rule" by S. C. Kohs of Reed College, 749–750. Volume 9, January 4, 1919: "Mathematics in general education" by C. N. Moore, 31–33—February 15: "The status of mathematics and mathematical instruction during the colonial period" by Herbert Kimmel, 195–202. [Mr. Kimmel is director of the Training School, State Normal College, Bowling Green, Ohio, and his article is the result of a brief study made while taking a course (at the University of Chicago) on American Colonial Education by Dr. Marcus W. Jernegan.]

SCHOOL REVIEW, Chicago, volume 26, nos. 8–10, October–December, 1918: "A preliminary attempt to devise a test of the ability of high-school pupils in the mental manipulations of space relations" by H. N. Irwin, 600–605, 654–670, 759–772.

SCHOOL SCIENCE AND MATHEMATICS, volume 18, 1918, October: "Map scales" by R. M. Mathews, 596–600; "Graphical algebra as applied to functions of the nth degree" by F. E. Nipher, 603–605; "Mock trial of B versus A or solving a personal equation by the judicial process"[1] (Mr. Stephen Leacock's story "A, B, C") by Kathryn McSorley, 611–621; "A graphical representation of approximations for square root" by O. Dunkel, 621–625; "Two methods of locating the German super gun," by H. F. MacNeish, 626–628; Problem department, 658–661—November: "Some group theory" by G. A. Miller, 675–680; "Bolshevik multiplication" by H. J. R. Twigg, 698; "Mathematics and education" by L. E. Mensenkamp, 709–711;

[1] Presented at a meeting of The Mathematics Club of Hunter College, December 13, 1917.

Problem department, 756–758—No. 9, December: "On the history of 'Playfair's parallel postulate'" by F. Cajori, 778–779; "Making algebra feed the allies" by F. M. Rich, 811–819; "A model of supplementary trihedral angles," 846–848; Problems and solutions, 849–853—Volume 19, no. 1, January, 1919: "Mathematics and anti-mathematics" by C. E. White, 29–37; "The teaching of first year algebra" by W. W. Stradler, 38–44; "Adopt the metric system" by H. N. Kauffman, 82–84; Problems and solutions, 85–88—No. 2, February: "The development of the function concept" by L. L. Dines, 99–110 [read at the annual meeting of the Mathematical and Physical Society of Saskatchewan, April 2, 1918]; "Periodic decimal fractions" by M. O. Tripp, 110–113; "An elementary study of physical units" by S. Waddell, 114–118; "Determination of the refracting angle of a prism, and the angle of minimum deviation" by C. H. Skinner, 119–122; Problems, 186–189.

SCIENCE, volume 49, 1919, January 24: "Professor Ludvig Sylow" by G. A. Miller. [Quotation: "In view of the general interest in the retirement of university professors at sixty-five it may be worth noting that Sylow was appointed professor of mathematics in the University of Christiania after reaching the age of sixty-five years. While various other noted European mathematicians were called to university positions after they had spent years in teaching in secondary institutions, Sylow was perhaps the only one among them who devoted forty years to teaching in a secondary institution before securing a university chair."].

SCIENCE PROGRESS, London, volume 13, 1918, July: "The 34 supermagic square" by F. J. Anderson, 86–96 [Reprinted in *Current Opinion*, volume 65, pp. 376–378, December, 1918, and in the *Scientific American Supplement*, volume 87, 44–45, January 18, 1919]; Reviews by P. E. B. Jourdain of Forsyth's *Theory of functions of a complex variable* (third edition), F. S. Carey's *Infinitesimal Calculus*, and Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe*, 147–150—October: "Operative algebra: operative involution" by R. Ross, 288–298; "Problems of arrangement of an infinite class" by P. E. B. Jourdain, 299–304; Reviews by P. E. B. Jourdain of Woods and Bailey's *Analytic Geometry and Calculus*, Hancock's *Theory of maxima and minima* and Picard's *Les sciences mathématiques en France depuis un demi-siècle*; on pages 176–177 Jourdain comments as follows in connection with the AMERICAN MATHEMATICAL MONTHLY: "A very interesting feature of this MONTHLY is the accounts of the really great work that is being done in most American universities in the formation of undergraduate mathematical clubs, in which discussions take place and lectures are given—often by the students themselves—to show that mathematics is a living thing with a history and even human interests. Particularly valuable are the well-documented suggestions of suitable topics. Naturally D'Arcy Thompson's recent book *On Growth and Form* (Cambridge, 1917) is suggestive in this connection . . . and on allied subjects we have excellent notes in the last three of the following list of suggestions that have been made: Rhind papyrus, geometrography, arithmetical prodigies, Ptolemy's theorem, paper folding, women as mathematicians and astronomers, binary arithmetic, the logarithmic spiral, golden section, and a Fibonacci series . . . "—January, 1919: "Physical relativity hypotheses old and new" by G. W. de Tunzelman, 475–482; "Isosceles trigonometry: suggested additions to the trigonometrical ratios" by R. Ross, 485–486; [Quotations: "At present trigonometry is based entirely upon the right-angled triangle; but a useful addition can be founded upon the isosceles triangle. Thus, for the solution of triangles, any right-angled triangle may be divided into two isosceles triangles by drawing a straight line from the right angle so as to bisect the opposite side; and hence any triangle may be discussed in terms of four isosceles triangles. For this purpose it is useful to add two trigonometrical ratios to the list in use. If A is the angle between the two equal sides of an isosceles triangle, we may denote by bas A the ratio of the base to one of the sides; and if sA is the supplement of the angle A, we may denote bas $(\pi - A)$ by bas sA. We may then obtain the following easily verified relations: $\cdots$ $\text{bas}^2 A + \text{bas}^2 sA = 4 \cdots \text{bas}\, \pi/2 = \sqrt{2} \cdots (\text{bas}\, s\theta + i\, \text{bas}\, \theta)^n = \text{bas}\, sn\theta + i\, \text{bas}\, n\theta \cdots$." P. E. B. Jourdain adds the following note: "It will be observed that the new functions have a periodicity of 4π; and hence, to a value of bas x (say) between θ and π, only one value of x belongs, whereas to $\sin x$ (say) two values belong. Thus it seems that these interesting functions may be also of practical importance in avoiding 'the ambiguous case' in the solution of triangles"]. Review by D. O. Wood of Singer's *Studies in the history of science* (Oxford, 1917); Reviews, by H. S. Jones, of Gill and Merrifield's *Navigation and Nautical Astronomy* (London, 1918), Arrhenius's *The Destinies of the Stars* (London, 1918), and of Plummer's *Introductory Treatise on Dynamical Astronomy* (Cambridge, 1918), 490–493; Review by L. Silberstein of Eddington's *Report on the Relativity of gravitation* (London, 1918), 493–494.

SCIENTIA, volume 24, 1918, December: Review by A. Mieli of *Studies in the History and Method of Science* edited by C. Singer (Clarendon Press, 1917), 471–473; review by G. Loria of *An Introduction to the Algebra of Quantics* by E. B. Elliot (second edition, Clarendon Press, 1913), 473–475—Volume 25, 1919, January: "Influence grecque dans le développement des mathématiques hindoues" by G. R. Kaye, 1–14; Reviews, by P. E. B. Jourdain, of Russell's *Mysticism and logic*, Keyser's *Human worth of rigorous thinking*, and Huntington's *The continuum, and other types of serial order*, 62–66.

SCIENTIFIC MONTHLY, volume 8, February, 1919: "Motives for the cultivation of mathematics," by R. D. Carmichael, 160–178. The last of eight verses with which the article closes is as follows:

'Tis this the watchful, keen Researcher sees:
    It guides him to a bounded field of thought;
It teaches him the need of new degrees
    Of power by which his conquests may be brought
    To bear on widest realms with blessing fraught.
A zeal arises in his inmost heart,
    A consecration by deep purpose wrought
Absorbs his strength, his life he sets apart
To bless mankind by streams of truth which from him start.

THE SEWANEE REVIEW, Sewanee, Tenn., volume 26, 1918, October–December: "Archibald Henderson" by E. Markham, 468–473. [Quotations: "With occasional flights out into the world, Mr. Henderson has found time for scientific, literary, and cultural activities that almost make us think that in his case time keeps no count of hours. Mathematical and scientific societies of America and Europe know Dr. Henderson for his learned papers in their own fields. His treatise, *The Twenty-seven Lines on the Cubic Surface*, which was published by Cambridge University, England, has been highly lauded at Oxford and Cambridge, and received encomiums from mathematicians in Denmark, Italy, and Japan."

". . . But Archibald Henderson's greatest biography, and one of the super-biographies of the world, is his oceanic presentation of that insurgent and cryptic personality George Bernard Shaw. The publication of this comprehensive work raised a shout in the literary world. It was a huge life study of a living, changing man, hewn out against a background of the historic, literary and social events of his time. It was a work of large significance not only in tracing the currents of modern thought, but also in giving the pulse and pressure of this Shavian force that is so mighty a spiritual energy for moulding human opinion. This work nailed the name of Archibald Henderson high on the pillar of literary achievement. Bernard Shaw himself said: 'You are a genius, because you are somehow susceptible to the really significant and differentiating traits and utterances of your subject.'"]

THE TÔHOKU MATHEMATICAL JOURNAL, volume 14, nos. 3–4, November, 1918: "Sets of independent postulates concerning equality and inequality (theory of three undefined relations)" by K. Yoneyama, 171–283; "On Fourier constants" by K. Ogura, 284–296; "Bemerkungen über die Beziehung zwischen Erfahrung und Geometrie" by H. Tanabe, 297–304; "On the Dajutu or the arithmetic series of higher orders as studied by the wasanists" by K. Yanagihara, 305–324; "On the roots of the algebraic equation of the form $f + k_1 f' + k_2 f'' + \cdots + k_n f^{(n)} = 0$" by Y. Uchida, 325–327; "On some algebraic equations whose roots are all real and distinct" by Y. Okada, 328–333; "On some algebraic equations having real roots only" by T. Hayashi, 334–339; "On inversion with respect to a conic" [in Japanese] by S. Fukazawa, 340–349.

WASHINGTON UNIVERSITY STUDIES, scientific series, volume 5, no. 2, January, 1918: "On the F_1-functions of the calculus of variations" by P. P. Rider, 97–100—Volume 6, no. 1, July, 1918: "Necessary conditions for the reality of all the roots of an algebraic equation" by Otto Dunkel, 13–19—no. 2, January, 1919: "A geometric treatment of the exponential function" [1] by O. Dunkel, 33–38.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT ALLER SCHULGATTUNGEN, volume 49, Doppel-Heft 8–9, September, 1918: "Das 'Petersburger Problem' der Wahrscheinlichkeitsrechnung" by O. Knopf, 233–242; "Der Taylorsche Lehrsatz im Unterricht" by C. Andriessen, 242–247; Das Bilden von kubischen Gleichungen mit vorgeschriebenen Eigenschaften" by E. Haentzschel, 248–254; "Zum Nachweis der Zentrifugalkraft

[1] Read before the Missouri section of The Mathematical Association of America at the first regular meeting held in St. Louis, Mo., November 18, 1916.

der Erddrehung "by W. Brunner, 254–262; "Max Simon zum Gedächtnis" (mit Bild im Text) by W. Lorey, 268–271. [A more elaborate "Nachruf" by the same author is in *Leopoldina*, 1918, no. 2. Max Simon was born in 1844 and died January 15, 1918. He was Oberlehrer at the Lyceum in Strasbourg for over 40 years[1] and "Honorarprofessor" at the university since 1903. His most successful book was probably his *Didaktik und Methodik des mathematischen Unterrichts*. The long list of his writings in elementary mathematics and mathematical history includes the mathematical articles A–L, as well as the biographies of all mathematicians, in the *Konversationslexikon* of Brockhaus. He was also the author of the mathematical articles A–R in the fifth edition of "Meyer"].—Doppel Heft 10–11, November, 1918: "Ueber isoperimetrische Probleme in der Schule und in der Forschung" by W. Lorey, 281–293; "Nochmals die Hessesche Normalform" by K. Doehlemann, 293–299; "Ueber eine algebraische Behandlungsweise des regulären Siebzehnecks" by H. Wolff, 299–303; "Die Siebenzehn-Teilung des Kreises in elementargeometrischer Herleitung" by R. Lohnstein, 303–312; "Zur Ortsbestimmung eines Fesselballons" by A. Witting, 312–317; "Der räumliche pythagoreische Lehrsatz" by H. Wieleitner, 321–322. [Wieleitner's note deals with the history of the theorem: If a tetrahedron has each of its face angles at one vertex a right angle, the sum of the squares of the areas of the faces about the vertex is equal to the square of the area of the face opposite the vertex. Wieleitner states that this theorem was first given in *Johann Faulhabers Ulmensis Miracula Arithmetica*, Augspurg, M.DC.XXII, chapter 45, pp. 73–75. Faulhaber refers to the theorem as "ein Newe Geometrische Invention welche auss der Zahl 666 (Apocal. in 13. Cap.) Calculirt und Demonstrirt." It is pointed out that the theorem in its generality was known to Descartes who discussed it in his *Cogitationes privatæ* (written 1619–21, but first printed in 1908) where it is stated in the form: In tetraedro rectangulo, basis potentia æqualis est potentijs trium facierum simul. It is known that Faulhaber was in personal touch with Descartes about the time that these *Cogitationes* were written.]

AMERICAN DOCTORAL DISSERTATIONS.

J. W. CAMPBELL, 1889– , *Periodic solutions of the problem of three bodies in three dimensions*. Pp. 43–84. [Reprinted from *The Proceedings of the London Mathematical Society*, 1917. (Chicago, 1915.)

A. M. HARDING, *On certain loci projectively connected with a given plane curve*. 38 pp. [Reprinted from *Giornale di matematiche di Battaglini*, 1916.] (Chicago, 1916.)

W. L. HART, 1892– , *Differential equations and implicit functions in infinitely many variables*. Pp. 125–160. [Reprinted from *Transactions of the American Mathematical Society*, 1917.] (Chicago, 1916.)

MARY G. HASEMAN, *On Knots, with a census of the amphicheirals with twelve crossings*. [Reprinted from the *Transactions of the Royal Society of Edinburgh*, volume 52, 1917]. Edinburgh, Neill and Co., 1918. Pp. 235–255 + 1 plate. 4to. (Bryn Mawr, 1916).

F. M. MORRISON, 1871– , *On the relation between some important notions of projective and metrical differential geometry*. Pp. 199–221. [Reprinted from the *American Journal of Mathematics*, 1917.] (Chicago, 1913.)

MARY E. WELLS, *On inequalities of certain types in general linear integral equation theory*. Pp. 163–184. [Reprinted from the *American Journal of Mathematics*, 1917.] (Chicago, 1915.)

MEETING OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America met at Johns Hopkins University, Baltimore, Maryland, January 18, 1919. Among those in attendance were the following members: Oscar S. Adams, U. S. Coast and Geodetic Survey; H. G. Avers, U. S. Coast and Geodetic Survey; Clara L. Bacon, Goucher College; G. R. Clements, U. S.

[1] *Cf.* S. WIRZ, *Der mathematische Unterricht an den höheren Knabenschulen sowie die Ausbildung der Lehramtskandidaten in Elsass-Lothringen*. (IMUK Abhandlung), Leipzig, Teubner, 1911.

Naval Academy; A. Cohen, Johns Hopkins University; Alexander Dillingham, U. S. Naval Academy; H. C. Gossard, U. S. Naval Academy; W. M. Hamilton, U. S. Nautical Almanac Office; Frank Morley, Johns Hopkins University; F. D. Murnaghan, Johns Hopkins University; R. E. Root, U. S. Naval Academy; C. A. Shook, U. S. Naval Academy; Clara E. Smith, Wellesley College (Goucher, 1918–1919); H. Ivah Thomsen, Johns Hopkins University; G. F. Winslow, Jr., U. S. Coast and Geodetic Survey; L. S. Hulburt, Johns Hopkins University.

The program consisted of a single session in the afternoon, and contained the following papers:

1. Content of a course in analytic geometry. By DR. G. R. CLEMENTS, U. S. Naval Academy. Discussion led by PROFESSOR CLARA L. BACON, Goucher College.

2. Lambert's mapping. By PROFESSOR FRANK MORLEY, Johns Hopkins University.

3. The teaching of the subject of limits. By PROFESSOR CLARA E. SMITH, Wellesley College.

4. The polyconic projection and the quadrillage associated therewith. By MR. O. S. ADAMS, U. S. Coast and Geodetic Survey.

After the program the members present were guests of Johns Hopkins University at the Hopkins Club, where a very enjoyable supper was served.

R. E. ROOT, *Secretary.*

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I. [*1918*, 33–34].

For 1918–19 the club is to be guided by

Chairman: Professor Nathaniel F. Davis;

Committee on Program: Professor Roland G. D. Richardson, Winona M. Perry Gr., Frances W. Wright '19, Harley F. Carey '20, Robert B. Lindsay '20;

Committee on Arrangements: Professor Theodore H. Brown, Rachel T. Easterbrooks '20, Clarence R. Adams Gr., Everett L. Sweet '21.

The following meetings have been held.

November 30, 1918: "The method of solving certain problems in the Ahmes papyrus" by Arnold B. Chace, Chancellor of the University.

February 14, 1919: "Archimedes" by Frances W. Wright '19; "Two methods of locating the German super-gun"[1] by Daniel B. Whitford '20; "The cattle problem of Archimedes" by Frances M. Merriam '20; "The logarithmic spiral" by Robert B. Lindsay '20.

[1] Report on H. F. MacNeish's article in *School Science and Mathematics*, October, 1918 (Vol. XVIII, pp. 626–628).

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City [*1918*, 187–188].

So far as the editor knows this club is the first one to make use of special printed stationery for its correspondence. According to the letter head the officers for the year 1918–19 are as follows: President, Anita Rosenthal '19; vice-president, Kathryn McSorley '19; secretary, Louise Biehl '19; treasurer, Miriam Werner, instructor in mathematics.

The spirit of war service dominated the work of the Club throughout the year 1918. Most of the members were volunteer workers aiding the Legal and Local Draft Boards throughout the city and promoting the Hunter College drives for Liberty Loans, Red Cross, Y. W. C. A. and other organizations. A number took courses in the War Service Training School for Women and about a dozen worked as farmerettes.

The programs for the year were as follows.

March: "The need of evaluating one's self so as to render the most effective service" by Professor Emma M. Requa; "Mr. and Mrs. Ratio One to Two demonstrating practical mathematics" by Marion Graham '18 and Ellen Raymond '18; social meeting.

April: "Commercial education of the present day" by Mr. Alfred Sommerfield of Washington Irving High School, New York.

May: "High School mathematics as an aid to the interpretation and appreciation of life about us" by Professor Charles B. Upton of Teachers' College, Columbia University.

June: General business meeting; election of officers.

September: The social meeting was postponed on account of war activities.

October: Social meeting.

November: "The common place book" by Charles B. Walsh of Ethical Culture School, New York.

THE MATHEMATICAL AND ASTRONOMICAL CLUB OF SWARTHMORE COLLEGE, Swarthmore, Pa. [*1918*, 135].

The officers of the club for the year 1917–18 were as follows: First semester—president, Robert S. Blau '18; vice-president, Ethelwyn Bower '18; secretary, Gladys Pell '20. Second semester—president, John Trimmer '18; vice-president, Dorothy A. Johnson '18; secretary, Charlotte Moore '20.

The programs for the first six meetings of the year 1917–18 were given in an earlier number of the MONTHLY. The programs for the remaining meetings of the year were as follows:

December 18, 1917; "The depression range finder and its use in the present war" by John Trimmer '18; "The Barr and Strand range finder" by Ewing T. Corson, '18; "The use of a range finder for objects invisible from the place where the gun is located" by Robert S. Blau '18.

February 5, 1918: "The astrology of casting a horoscope" by Opal M. Robinson '18; "The history of mathematics" by Albert N. Nelson '18; "De Moivre's theorem and its applications" by Lena Clark '20; "Graphic methods of solving equations" by Professor John H. Pitman.

February 19: "Rifling a cannon" by Robert S. Blau '18; "The discovery of Neptune" by Boyd J. Brown '21; "Scales of notation" by Elizabeth Trorer '19; "Intra-Mercurial planets" by Professor John A. Miller.

March 5: "Methods used in passing from the properties of rectilinear figures to circular figures" by Ethelwyn Bower '18; "Properties of magic squares" by Albert N. Nelson '18; "Principles and applications of the planimeter" by Harry Yardley '19; "The use of the planimeter in finding the displacement of a ship" by John Trimmer '18.

March 19: "The solar eclipse of 1905" by Professor Miller; "The problem of mounting long focus cameras for photography" by Margaret E. Powell '19; "Theories of the formation of the corona" by Caroline H. Smedley Gr.

April 2: "The discriminant and its relation to higher plane curves" by Helen Deputy '18; "Structure of powder grains" by Frank Fetter '20; "Problems of gunnery" by John Trimmer '18; "The Gregorian calendar" by Reverend Walter A. Matos.

April 19: "The history of algebra," an illustrated lecture by Professor Louis C. Karpinski of the University of Michigan.

THE MATHEMATICAL AND PHYSICAL SOCIETY OF THE UNIVERSITY OF TORONTO, Toronto, Ontario [*1918*, 229–231].

Regular meetings of this club are held on alternate Thursdays at 4:15 p. m. The officers for the year 1918–19 are as follows: Honorary president, Professor Eli F. Burton; president, William W. Shaver '19; vice-president, Percy Lowe '20; secretary-treasurer, William S. Vaughan '20; corresponding secretary, Ila B. Giles '19; representatives of classes: Lily M. Floody '19, Arthur J. Sonley '20, Nora E. Gray '21, Dorothy G. Gavin '22.

The following programs are announced for the current year:

November 21, 1918: "The darkest year in British history" by Professor Alfred Baker.

December 5: "Mathematical puzzles" by Professor John Satterly; "The moon" by Franklin B. Keachie '19.

December 17: Social evening at 8 p. m.

January 9, 1919: "New Zealand" by Professor John C. Fields; "My first impressions of the course" by Mattie Levi '21.

January 23: Graduates' meeting. "Periodic precipitations" by Alice W. Foster Gr.; "The ancient scientists" by Mabel C. Child '18.

January 28: Skating party.

February 6: Open meeting at 8 p. m.

February 20: Debate between First and Second Year Students.

March 6: "The most important experiment in fourth year physics" by Mary I. Mackay '19; "Mathematical recreation" by Everett O. Hall '19.

March 20: "Radium: its discovery and use" by Raymond C. Dearle Gr.; annual elections.

THE UNDERGRADUATE MATHEMATICS CLUB OF THE UNIVERSITY OF WASHINGTON, Seattle, Washington.

The information given below concerning this club was furnished by Assistant Professor Eric T. Bell of the University of Washington.

The club was in existence for some ten years and had a membership of about twenty-five. Meetings were held twice a month with an average attendance of about fifteen. The outbreak of the war put an end to the club's activities, since all but two of its members were taken in the draft or in other service. The club is soon to be reorganized.

The club was managed entirely by the students without any interference or direction whatever from the faculty and the results amply justified this course. Members of the faculty were invited to attend the meetings of the club and occasionally did so. On rare occasions, when the students had not had time to prepare papers of their own, they asked a member of the faculty to give an account of some branch of mathematics not in the undergraduate courses; for example, mathematical crystallography, groups, higher arithmetic, applications of mathematics to biology, etc. Meetings were scheduled to last for an hour and a half to two hours but sometimes continued for as long as three hours.

Detailed information concerning programs is lacking since the records kept by the secretaries disappeared when the last secretary left the university at the outbreak of the war. Some of the programs given, however, were as follows.

"Ten British mathematicians." A review of Alexander Macfarlane's book (1916) of that title—"Non-euclidean geometry." Presented and discussed at several meetings by members who had studied the subject. Bonola's (1912) and Manning's (1901) books were used as sources—"Geometry of four dimensions." A presentation during several meetings of Manning's book (1914) and a sketch of the circle-representation as given in Weber-Wellstein (1905)—"Hilbert's proof of the transcendence of π"—"Continued fractions and 'Pell's equation'"—"Formal implication."

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2762. Proposed by N. P. PANDYA, Amreli, India.

$ABCD$ is a cyclic quadrilateral inscribed in an ellipse. $AB = 2BC$ and $CD = 2DA$. Find the eccentricity of the ellipse in terms of the sides of the quadrilateral.

2763. Proposed by C. N. SCHMALL, New York City.

Show that the equation, $ky - 2k^{1/3}a^{2/3}x + x^2 = 0$, where k is a variable parameter, represents a family of parabolas passing through a fixed point, and all having the same areas, comprised between the curve and the x-axis.

Show, also, that the envelope of the family is the rectangular hyperbola whose equation is $xy = 2^5a^2/3^3$.

2740. Proposed by E. H. CLARKE, Hiram College.

If the coefficients of $(a - b)^k$ (where k is a positive integer) be multiplied term by term by the nth powers (n being zero or a positive integer), of the terms of any arithmetic progression with common difference $d \neq 0$, the sum of the products will vanish if $n < k$; will be $(-d)^k(k!)$ if $n = k$; and, if $n = k + 1$, will be the product of this last result and the sum of the terms of the arithmetic progression.

2765. Proposed by A. M. HARDING, University of Arkansas.

ABC is an equilateral triangle. A point D is taken in BC such that BD is $\frac{1}{3}$ of BC and E is taken in CA such that CE is $\frac{1}{3}$ of CA. If the lines AD and BE intersect at O, show that OC is perpendicular to AD.

2766. Proposed by N. P. PANDYA, Amreli, India.

Is it possible to find a harmonic series whose terms are positive integers such that the product of the first, second, seventh, and eighth terms is equal to the product of the third, fourth, fifth, and sixth terms?

2767. Proposed by W. W. JOHNSON, U. S. Naval Academy.

Let the complex quantities p, q, and r satisfy the relation $p^2 + q^2 + r^2 = 0$; prove that the corresponding vectors OP, OQ, and OR are such that if any two of them are taken as conjugate semi-diameters of an ellipse, the third lies on the minor axis and its length is the distance from the center to either focus.

2768. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given the center, a focus, and a point of a conic, construct geometrically the circle of curvature at the point.

2769. Proposed by B. J. BROWN, Kansas City, Mo.

Expand in powers of x as far as x^2 the function $\dfrac{\cosh \lambda x}{\cosh \lambda} - \dfrac{x \sinh \lambda x}{\sinh \lambda}$ in which λ is a positive constant.

Prove that, if $\lambda \tanh \lambda > 2$, the function has only one maximum value for $x > 0$ and that the value of x for which the maximum occurs is less than 1. India Civil Service. 1912.

2770. Proposed by A. M. HARDING, University of Arkansas.

Solve the differential equation

$$\frac{d^2x}{dt^2} - 2(\mu t + \lambda)\frac{dx}{dt} + (1 - \mu)x = 0,$$

where λ and μ are constants.

2771. Proposed by GEORGE PAASWELL, New York City.

A circle is revolved through an angle of 90° about a vertical chord which does not pass through the center of the circle. Taking the origin at the lower extremity of the chord, the z-axis along the chord, and the x- and y-axes in the boundary planes, pass a plane through the x-axis making a given angle with the xy-plane. Determine the portion of the area of the surface above the plane and between the xz- and yz-planes.

NOTE.—This is a restatement of calculus problem 430, a solution of which appeared in this MONTHLY, February, 1913. As the solution there given is not according to the interpretation intended by the Proposer, we are reprinting the problem in this slightly revised form—EDITORS.

2772. Proposed by HARRY LANGMAN, New York City.

Given $1 = (-\frac{1}{2} + x)^r = (-\frac{1}{2} - x)^r$, where r is an integer. Prove that r is a multiple of 3. In general, if

$$1 = \left(\cos\frac{2\pi}{m} + x\right)^r = \left(\cos\frac{2\pi}{m} - x\right)^r,$$

where r and m are integers, prove that r is a multiple of m.

SOLUTIONS OF PROBLEMS.

2689 [April, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

Show that the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \varphi + \theta)}$$

is $y_1 = (\cos\theta - \rho)/(\cos\theta + \rho)$ where $\rho = \sqrt{\sin^2\varphi - \sin^2\theta}$.

This problem was suggested to the proposer by a professor of civil engineering, and has important applications in the theory of conjugate stresses.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \varphi + \theta)}. \tag{1}$$

Taking the logarithm of (1), differentiating and setting the result equal to zero we get

$$\cot x - \tan (x + \varphi) + \tan (x + \theta) - \cot (x + \varphi + \theta) = 0 \tag{2}$$

or

$$\cot x - \cot (x + \varphi + \theta) = -\tan (x + \theta) + \tan (x + \varphi);$$

whence

$$\frac{\sin (\varphi + \theta)}{\sin (\varphi - \theta)} = \frac{\sin x \sin (x + \varphi + \theta)}{\cos (x + \varphi) \cos (x + \theta)}. \tag{3}$$

Or writing (2) in the form

$$\cot x + \tan (x + \theta) = \tan (x + \varphi) + \cot (x + \varphi + \theta)$$

we get

$$\frac{\cos\theta}{\sin x \cos (x + \theta)} = \frac{\cos\theta}{\cos (x + \varphi) \sin (x + \varphi + \theta)};$$

or

$$\frac{\cos (x + \varphi)}{\cos (x + \theta)} = \frac{\sin x}{\sin (x + \varphi + \theta)}. \tag{4}$$

(4) in (3) gives

$$\frac{\cos (x + \varphi)}{\sin x} = \sqrt{\frac{\sin (\varphi - \theta)}{\sin (\varphi + \theta)}};$$

whence

$$\cot x = \tan\varphi + \sec\varphi \sqrt{\frac{\sin (\varphi - \theta)}{\sin (\varphi + \theta)}}. \tag{5}$$

By (4) and (1), we may write

$$y_1 = \frac{\sin^2 x}{\sin^2 (x + \varphi + \theta)}$$

so that by (5)

$$\frac{1}{\sqrt{y_1}} = \cos (\varphi + \theta) + \tan\varphi \sin (\varphi + \theta) + \sec\varphi \sqrt{\sin (\varphi + \theta) \sin (\varphi - \theta)} = \frac{\cos\theta + \rho}{\cos\varphi}.$$

$$\therefore\ y_1 = \frac{\cos^2\varphi}{(\cos\theta + \rho)^2} = \frac{\cos^2\theta - \rho^2}{(\cos\theta + \rho)^2} \equiv \frac{\cos\theta - \rho}{\cos\theta + \rho}.$$

To prove that we have a maximum we find:

$$\frac{dy}{dx} = \frac{\cos\theta}{\cos (x + \theta) \sin (x + \varphi + \theta)} \left\{ \frac{\cos (x + \varphi)}{\cos (x + \theta)} - \frac{\sin x}{\sin (x + \varphi + \theta)} \right\}$$

$$= \frac{\cos\theta}{\cos^2 (x + \theta) \sin^2 (x + \varphi + \theta)} \{\sin\varphi \cos (2x + \varphi + \theta) + \sin\theta\},$$

which shows that we have either a maximum or a minimum when the brace equals zero, since it is the only variable factor of odd power. Choosing the least value of x that will make the brace zero we have that $\cos (2x + \varphi + \theta)$ decreases as x increases so that the sign of dy/dx changes from positive to negative as x passes through this critical value.

2690 [April, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

Find the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \beta + \theta)}. \tag{1}$$

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Taking the logarithm of both sides, differentiating, and setting the result equal to zero we get

$$\cot x - \tan (x + \varphi) + \tan (x + \theta) - \cot (x + \beta + \theta) = 0 \tag{2}$$

and from the two forms of this

$$\cot x - \cot (x + \beta + \theta) = \tan (x + \varphi) - \tan (x + \theta)$$

and

$$\cot x + \tan (x + \theta) = \cot (x + \beta + \theta) + \tan (x + \varphi)$$

we get

$$\frac{\sin (\beta + \theta)}{\sin (\varphi - \theta)} = \frac{\sin x \sin (x + \beta + \theta)}{\cos (x + \varphi) \cos (x + \theta)} \tag{3}$$

and

$$\frac{\cos (x + \varphi)}{\cos (x + \theta)} = \frac{\cos (\beta + \theta - \varphi)}{\cos \theta} \cdot \frac{\sin x}{\sin (x + \beta + \theta)}. \tag{4}$$

By (3) and (4), we have

$$\sqrt{\frac{\cos (\beta + \theta - \varphi) \sin (\varphi - \theta)}{\cos \theta \sin (\beta + \theta)}} = \frac{\cos (x + \varphi)}{\sin x} = \cot x \cos \varphi - \sin \varphi.$$

Hence,

$$\cot x = \tan \varphi + \sec \varphi \sqrt{\frac{\cos (\beta + \theta - \varphi) \sin (\varphi - \theta)}{\cos \theta \sin (\beta + \theta)}}. \tag{5}$$

(4) and (1) give, where y_1 is the maximum value of y,

$$y_1 = \frac{\cos (\beta + \theta - \varphi)}{\cos \theta} \cdot \frac{\sin^2 x}{\sin^2 (x + \beta + \theta)};$$

whence

$$\frac{1}{\sqrt{y_1}} = \sqrt{\frac{\cos \theta}{\cos (\beta + \theta - \varphi)}} \{\cos (\beta + \theta) + \sin (\beta + \theta) \cot x\}$$

and eliminating $\cot x$ by (5), we have

$$y_1 = \cos^2 \varphi / \{\sqrt{\cos \theta \cos (\beta + \theta - \varphi)} + \sqrt{\sin (\varphi - \theta) \sin (\beta + \theta)}\}^2.$$

As in 2689 we may prove that y_1 is a maximum as found for the case $\beta = \varphi$ from which we may infer from the manner that β enters into the expression that it is a maximum for all values of β.

2692 [April, 1918]. Proposed by J. L. RILEY, Stephenville, Texas.

A cube is cut at random by a plane, what is the chance that the section is a hexagon?

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Let the faces of the cube be $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$. Let the plane be

$$lx + my + nz = p,$$

where $n = \cos \theta > 0$, $l = \sin \theta \cos \phi$, $m = \sin \theta \sin \phi$. The normal from the origin is equally likely to have any direction, and, when the direction is fixed, equally likely to have any length. Since $\sin \theta d\theta d\phi$ is an element of the solid angle traced by the normal, and dp an element of its length, the relative probability of the plane lying within given limits is $\iiint \sin \theta dp d\theta d\phi$. On account of the symmetry of the cube, it may be assumed that the normal through O to the plane meets the surface of the cube within the triangle (0, 0, 1), (1, 0, 1), and (1, 1, 1); so that $n > l > m > 0$, that is, $0 < \phi < \pi/4$, $0 < \theta < \tan^{-1} \sec \phi$.

The distances from the plane, with regard to sign, of the vertices $(1, 1, 1)$, $(1, -1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$ are $l+m+n-p$, $l-m+n-p$, etc.; and these are in descending order, except the two middle ones, which partake of the property only when $l+m>n$, that is when $\theta > \tan^{-1}(\cos\phi+\sin\phi)^{-1}$.

The plane will cut the cube when $l+m+n>p>-l-m-n$, and will cut it in a hexagon when $l+m-n>p>-l-m+n$. Hence, the required chance is

$$\frac{\int_0^{\pi/4}\int_{\tan^{-1}(\cos\phi+\sin\phi)^{-1}}^{\tan^{-1}\sec\phi}\int_{\cos\theta-\sin\theta(\cos\phi+\sin\phi)}^{-\cos\theta+\sin\theta(\cos\phi+\sin\phi)}\sin\theta\,dp\,d\theta\,d\phi}{\int_0^{\pi/4}\int_0^{\tan^{-1}\sec\phi}\int_{-\cos\theta-\sin\theta(\cos\phi+\sin\phi)}^{\cos\theta+\sin\theta(\cos\phi+\sin\phi)}\sin\theta\,dp\,d\theta\,d\phi}.$$

The integrations for p and θ having been effected, the inverse tangents in ϕ may be removed by integration by parts, after which the substitution $u=\tan\phi$ reduces the integrands to rational functions of u. The resulting value for the probability is $2/\sqrt{3}-(4\sqrt{2}/\pi)\tan^{-1}1/\sqrt{2}$, or $.0465\cdots$.

2693 [April, 1918]. Proposed by W. F. HARLOW, Portland, Oregon.

A cow is tethered with a rope, length l, to a peg on the opposite side of a wall, height h, the peg being at a distance a from the wall. Find the area over which the cow can graze.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Let the bottom of the wall be $x=z=0$; the top $x=0$, $z=h$; and the peg $(-a, 0, 0)$. The cow being at $(x, y, 0)$, the lengths of the two parts of the rope being u and v, and the point at which the rope crosses the wall being $(0, t, h)$, we must have, if the rope is taut,

$$u^2=a^2+h^2+t^2, \qquad v^2=x^2+(t-y)^2+h^2,$$

and $u+v=l$.

The solution of these equations in t, u, v is

$$t=\frac{yp\pm lr}{2(l^2-y^2)}, \qquad u=\frac{lp\pm yr}{2(l^2-y^2)}, \qquad v=\frac{lq\mp yr}{2(l^2-y^2)},$$

where $p=a^2+l^2-x^2-y^2$, $q=l^2+x^2-a^2-y^2$, and $r=\sqrt{p^2-4(a^2+h^2)(l^2-y^2)}$.

We require the conditions that u and v may be positive and t real.

From the obtuse triangle $(-a, 0, 0)$, $(0, 0, h)$, (x, y, h) it is clearly necessary that

$$l^2>a^2+h^2+x^2+y^2. \tag{1}$$

Other necessary conditions are that $p^2\geqq 4(a^2+h^2)(l^2-y^2)$, and (since $l>|y|$ and $|p|>r$) that $p>0$. From these two it follows that

$$x^2\leqq l^2+a^2-y^2-2\sqrt{a^2+h^2}\,\sqrt{l^2-y^2}, \tag{2}$$

which again implies that $(\sqrt{l^2-y^2}-\sqrt{a^2+h^2})^2\geqq h^2$, that is (with the help of (1)) that

$$y^2\leqq l^2-(\sqrt{a^2+h^2}+h)^2. \tag{3}$$

It remains to prove that (2) and (3) are not only necessary but sufficient. For this we need only observe that they make r real (and positive) $l>|y|$, $p>r$, and q positive. The lower signs will then give a suitable solution in t, u, and v.

The area to be computed is therefore

$$A=2\int_0^{\sqrt{l^2-(\sqrt{a^2+h^2}+h)^2}}\sqrt{a^2+l^2-y^2-2\sqrt{a^2+h^2}\sqrt{l^2-y^2}}\cdot dy.$$

Writing $\sqrt{l^2-y^2}=u$, $\sqrt{a^2+h^2}=c$, we get

$$A=2\int_{c+h}^{l}\frac{u\sqrt{(u-c)^2-h^2}}{\sqrt{l^2-u^2}}\,du.$$

To reduce to standard elliptic integrals, we may make

$$u=\frac{\lambda v+\mu}{v+1},$$

where $(\lambda - c)(\mu - c) = h^2$, $\lambda\mu = l^2$, $\lambda \geqq \mu$. If l is assumed $\geqq c + h$, as is natural, λ and μ are real, unequal, and greater than c. The result of the transformation is

$$A = 2(\lambda - \mu)\int_{\sqrt{\frac{\mu-c}{\lambda-c}}}^{\sqrt{\frac{\mu}{\lambda}}} \frac{\sqrt{(\lambda - c)v^2 - (\mu - c)}}{\sqrt{\mu - \lambda v^2}} \cdot \frac{\lambda v + \mu}{(v+1)^3}\, dv.$$

The final result will be simplified if we transform again, putting

$$v = \frac{\sqrt{\mu(\mu - c)}}{\sqrt{\mu(\lambda - c) - (\lambda - \mu)cw^2}},$$

which gives

$$A = 2c(\mu - c)\int_0^1 \frac{(3\lambda c + 3\mu c - 2\lambda\mu - 4c^2)\mu^2 + (\lambda\mu + 5\mu c - 3\mu^2 - 3\lambda c)\mu c w^2 + (\lambda - \mu)\mu c^2 w^4}{(\mu - cw^2)^3\sqrt{1 - w^2}\sqrt{\mu(\lambda - c) - (\lambda - \mu)cw^2}} \cdot w^2 dw$$

$$+ 2c(\mu - c)\sqrt{\mu(\mu - c)}\int_0^1 \frac{(\lambda + \mu - 4c)\mu + (\lambda - 3\mu)cw^2}{(\mu - cw^2)^3\sqrt{1 - w^2}} \cdot w^2 dw.$$

These integrals are respectively elliptic and circular. On making the necessary reductions, we find, after much calculation,

$$A = c(\mu - c)\sqrt{\frac{\mu}{\lambda - c}}\, F\left(\sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right) - c\sqrt{\mu(\lambda - c)}\, E\left(\sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right)$$

$$- \frac{c(\mu - c)(\lambda + \mu - c)}{\sqrt{\mu(\lambda - c)}}\, \Pi\left(-\frac{c}{\mu}, \sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right) + \frac{\pi}{2} \cdot c(\lambda + \mu - c).$$

2697 [April, 1918]. Proposed by H. S. UHLER, Yale University.

Show how to reduce the left-hand members of the following identities to their respective right members:

$$\sin^2(x + \tfrac{1}{2}y) - \sin(x + \tfrac{3}{2}y)\sin(x - \tfrac{1}{2}y) = \sin^2 y, \tag{1}$$

$$\sin(x + y)\sin(x + \tfrac{1}{2}y) - \sin x \sin(x + \tfrac{3}{2}y) = \sin\tfrac{1}{2}y \sin y, \tag{2}$$

$$\sin x \sin(x + \tfrac{1}{2}y) - \sin(x - \tfrac{1}{2}y)\sin(x + y) = \sin\tfrac{1}{2}y \sin y. \tag{3}$$

SOLUTION BY POLYCARP HANSEN, St. John's University, Collegeville, Minn.

The terms of the left-hand members can be expressed as follows:

(1) $$\sin^2(x + \tfrac{1}{2}y) - \sin(x + \tfrac{3}{2}y)\sin(x - \tfrac{1}{2}y)$$
$$= \frac{1 - \cos(2x + y)}{2} + \frac{1}{2}[\cos(2x + y) - \cos 2y] = \frac{1 - \cos 2y}{2} = \sin^2 y.$$

(2) $$\sin(x + y)\sin(x + \tfrac{1}{2}y) - \sin x \sin(x + \tfrac{3}{2}y) = -\tfrac{1}{2}[\cos(2x + \tfrac{3}{2}y) - \cos\tfrac{1}{2}y]$$
$$+ \tfrac{1}{2}[\cos(2x + \tfrac{3}{2}y) - \cos(-\tfrac{3}{2}y)] = \tfrac{1}{2}[\cos\tfrac{1}{2}y - \cos(-\tfrac{3}{2}y)] = \sin\tfrac{1}{2}y \sin y.$$

(3) $$\sin x \sin(x + \tfrac{1}{2}y) - \sin(x - \tfrac{1}{2}y)\sin(x + y) = -\tfrac{1}{2}[\cos(2x + \tfrac{1}{2}y) - \cos(-\tfrac{1}{2}y)]$$
$$+ \tfrac{1}{2}[\cos(2x + \tfrac{1}{2}y) - \cos(-\tfrac{3}{2}y)] = \sin\tfrac{1}{2}y \sin y.$$

Also solved by R. B. WILDERMUTH, JEROME J. JULIAN, KATHERINE S. ARNOLD, R. M. MATHEWS, H. L. OLSON, H. E. CARLSON, A. T. DINEEN, R. C. COLWELL, and ROGER A. JOHNSON.

2698 [April, 1918]. Proposed by WARREN WEAVER, Throop College of Technology, Pasadena, California.

An urn contains N balls, numbered from 1 to N. Of these n are drawn out and are arranged linearly according to the numbers on each. A certain ball is observed to be the kth in this line. What is the most probable number written on this ball?

I. Solution by Harry M. Roeser, Bureau of Standards, Washington, D. C.

The total number of selections of n balls from N balls is

$$U = \frac{|N}{|n\,|N-n}.$$

The least number that can be in the kth place is k. In the urn there are $(N-k)$ numbers greater than k and $(k-1)$ numbers less than k. The numbers less than k can be arranged in groups of $(k-1)$ each in 1 way. After this is done the numbers greater than k can be arranged in groups of $(n-k)$ numbers each in $\frac{|N-k}{|n-k\,|N-n}$ ways. Therefore the total number of sequences with k in the kth place is

$$u_0 = \frac{1 \cdot |N-k}{|n-k\,|N-n}$$

and by definition the probability that k will be in the kth place is u_0/U.

The next lowest number that can be in the kth place is $(k+1)$. The k numbers less than $(k+1)$ can be arranged in groups of $(k-1)$ numbers each in

$$\frac{|k}{|k-1} = k$$

ways. After this is done the $N-k-1$ numbers greater than $(k+1)$ can be arranged in groups of $(n-k)$ numbers each in $\frac{|N-k-1}{|n-k\,|N-n-1}$ ways. Therefore, the total number of sequences with $(k+1)$ in the kth place is

$$u_1 = \frac{|N-k-1}{|n-k\,|N-n-1} \cdot k$$

and the probability that $(k+1)$ will be in the kth place is u_1/U.

By similar reasoning it readily follows that the respective probabilities that k, $(k+1)$, $(k+2)$, $\cdots$ $(k+i)$, $\cdots$, are in the kth place are given by the successive terms of the series,

$$\frac{1}{U}[u_0 + u_1 + u_2 + \cdots u_i + \cdots] = \frac{|N-n\,|n}{|N}\left[\frac{|N-k}{|n-k\,|N-n} + \frac{|N-k-1}{|n-k\,|N-n-1} \cdot k\right.$$

$$\left. + \frac{|N-k-2}{|n-k\,|N-n-2} \cdot \frac{(k+1)\cdot k}{|2} + \cdots \frac{|N-k-i}{|n-k\,|N-n-i} \cdot \frac{(k+i-1)\cdots k}{|i} + \cdots\right]$$

and the number most probably in the kth place is given by adding to k the value of i corresponding to the largest term of this series.

In the above series the ratio of the term corresponding to $(i+1)$ to the term corresponding to i is

$$m = \frac{N-n-i}{N-k-i} \cdot \frac{k+i}{i+1}.$$

Setting $m \gtreqless 1$ we find by reduction of the inequalities that

$$i \gtreqless N\frac{(k-1)}{n-1} - k$$

and, therefore, the number most likely to be found in the kth place is given by the integral part of $(k+i+1)$ or the integral part of $N\frac{k-1}{n-1} + 1$.

If $N\frac{k-1}{n-1}$ is an integer there will be two successive numbers $N\frac{k-1}{n-1}$ and $N\frac{k-1}{n-1} + 1$ each equally likely to be found in the kth place and more likely to be found there than any other numbers in the urn.

II. Mr. C. F. GUMMER solved the problem in a similar manner and added:

It may be of interest to find the expectation. This will be

$$\sum_{r=k}^{r=N-n+k} \frac{\binom{r-1}{k-1}\binom{N-r}{n-k}}{\binom{N}{n}} r = \frac{k\,|n}{|N} f(N, n, k),$$

where r is the number on the kth ball, and

$$f(N, n, k) = \sum_{r=k}^{r=N-n+k} \binom{N-n}{r-k} \frac{|r}{|k} \frac{|N-r}{|n-k}.$$

From the relation

$$\binom{N-n}{r-k} = \binom{N-n-1}{r-k-1} + \binom{N-n-1}{r-k},$$

it follows that

$$f(N, n, k) = (n-k+1)f(N, n+1, k) + (k+1)f(N, n+1, k+1). \qquad (1)$$

Now $f(N, N, k) = 1$, being independent of k. Therefore, by (1), $f(N, N-1, k) = N+1$, also independent of k, and finally $f(N, n, k) = (N+1)N\cdots(n+2)$. Hence, the expectation for the kth ball is $k(N+1)/(n+1)$.

2699 [May, 1918]. Proposed by the late ROGER E. MOORE, University of Wisconsin.

Show that if $a_k^{(r)}$ denotes the kth term of an arithmetic progression of order r, and c_k denotes the kth binomial coefficient in the expansion of $(a-b)^n$, n being a positive integer,

$$s \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \qquad \text{if} \qquad n > r.$$

SOLUTION BY ELBERT H. CLARKE, Hiram College.

Let d_0 be the first term in the arithmetic progression and let $d_1, \cdots, d_r$ denote the initial difference of each order. Using the usual abbreviated notation for binomial coefficients, we write

$$a_k^{(r)} = \binom{k-1}{0} d_0 + \binom{k-1}{1} d_1 + \binom{k-1}{2} d_2 + \cdots + \binom{k-1}{r} d_r,$$

$$c_k = (-1)^{k-1}\binom{n}{k-1}$$

and

$$\sum_{k=1}^{n+1} c_k a_k^{(r)} = \sum_{i=0}^{r} d_i \sum_{k=1}^{n+1} (-1)^{k-1}\binom{k-1}{i}\binom{n}{k-1}.$$

Consider the inner sum. Since $\binom{k-1}{i} = 0$, for $k < i+1$,

$$\sum_{k=1}^{n+1} (-1)^{k-1}\binom{k-1}{i}\binom{n}{k-1} = \sum_{k=i+1}^{n+1} (-1)^{k-1}\binom{k-1}{i}\binom{n}{k-1},$$

and the latter expression easily becomes

$$\binom{n}{i} \sum_{k=i+1}^{n+1} (-1)^{k-1}\binom{n-i}{k-(i+1)}.$$

Now put $k-i-1 = t$ and we have

$$\binom{n}{i} \sum_{t=0}^{n-i} (-1)^{t+i}\binom{n-i}{t}.$$

But the expression under summation is simply $(-1)^i(1-1)^{n-i}$. Hence, the coefficient of every d_i is zero. Therefore,

$$\sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \qquad n > r.$$

NOTE.—This proof is good only for $n > r$. Numerical examples can easily be given to show that the formula is not true for $n \leqq r$.

Also solved by HORACE L. OLSON.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. J. N. VAN DER VRIES has resigned his position as professor of mathematics at the University of Kansas to continue his work as secretary of the central district of the Chamber of Commerce of the United States, with headquarters at Chicago.

Mr. G. H. CRESSE, previously instructor in the University of Michigan, has been appointed to an instructorship in mathematics at the U. S. Naval Academy at Annapolis; he was granted the degree of doctor of philosophy at the University of Chicago in December.

Mr. F. S. NOWLAN, of Bowdoin College, has been promoted to an assistant professorship of mathematics.

Dr. MARION B. WHITE, formerly of the Ypsilanti State Normal College, Michigan, is now professor of mathematics at Carleton College.

Dr. TOBIAS DANTZIG and Dr. G. A. PFEIFFER have been appointed instructors in mathematics at Columbia University.

Professor GRIFFITH C. EVANS is now scientific attaché to the American Embassy at Rome.

Dr. W. GROSS, of the University of Vienna, has been promoted to professor of mathematics.

Professor E. WIECHERT of Göttingen has been appointed professor of geodesy and geophysics at the University of Berlin.

Efforts are being made to establish a chair of mathematical physics at the University of Edinburgh in memory of the late Professor TAIT.

At the University of Strasbourg Professor RENÉ M. FRÉCHET, of the University of Poitiers, has been appointed professor of mathematics, and PIERRE WEISS, professor at the Polytechnikum, Zurich, professor of general physics.

CHARLES L. DOOLITTLE, professor emeritus of astronomy at the University of Pennsylvania, and director of the Flower Observatory, died on March 3, 1919, aged seventy-five years.

Mr. ROGER E. MOORE, son of Professor Ransom A. Moore of the Wisconsin College of Agriculture, and a contributor to the pages of this MONTHLY, died at Camp Taylor, Kentucky, in October, 1918.

FRANÇOIS DANIËLS of Nymwegen, Holland, professor of mathematics at the University of Fribourg, Switzerland, died on November 16, 1918, at the age of fifty-eight years. He was the author of *Essai de géométrie sphérique en coordonnées projectives* (Fribourg, 1907) and of articles in *L'Enseignement Mathématique*.

From the *Jahresbericht der Deutschen Mathematiker-Vereinigung* we learn of the deaths of the following mathematicians: Professor A. BENTELI, of the University of Bern, on November 10, 1917, in his seventieth year. Professor E. OTT, of the University of Bern, on November 17, 1917, in his seventieth year. Dr. ROBERT JENTZCH, of the University of Berlin, on March 21, 1918, fallen in battle. Professor M. B. WEINSTEIN, of Berlin, in his sixty-fifth year. Professor G. VERONESE, of the University of Padua, on July 17, 1917, in his sixty-third year.

Nature, for February 27, 1919, announces the death of ALEXANDER MICHAILOVITCH LIAPOUNOFF who held the chair of applied mathematics in the Petrograd Academy. "His later, and perhaps best-known, work dealt with the stability of the pear-shaped figure of a rotating mass of liquid, a problem of the first importance to theories of cosmogony. Poincaré had developed a method for the analytical discussion of the problem in 1901, but did not carry out the necessary calculations in detail, and so reached no definite conclusion. In 1902 Sir G. DARWIN announced that he had proved the pear-shaped figure to be stable, but this announcement was followed by a paper from Liapounoff in 1905, in which it was claimed that the pear-shaped figure was unstable. Liapounoff's work was distinguished by the combination of clear physical insight and masterly analytical skill." For somewhat more accurate statements in connection with the above see *Scientific Papers* by G. H. DARWIN, Volume 5, 1916, pp. xliii–xliv.

Professor H. E. BUCHANAN, of the University of Tennessee, has been employed in Y. M. C. A. work during the past year.

Professor JOSEPH ALLEN, of the College of the City of New York, and Professor W. H. METZLER, of the University of Syracuse, have gone to France on Army educational work.

Captain P. L. THORNE, assistant professor of mathematics at New York University, has recently returned to his university work. He served at the front in France with the Sixtieth Heavy Artillery regiment.

Captain A. L. UNDERHILL, of the University of Minnesota, has been appointed Commandant at the University of Grenoble in France, where several hundred American soldiers are taking courses while awaiting their opportunity to return home.

Professor EDWARD S. SMITH, of the department of mathematics at the University of Cincinnati, was, in addition to his regular duties, Acting Commandant of the Military Department from January to August, 1918, and Executive Secretary in charge of administrative matters in connection with the S. A. T. C. during the fall term.

The following reports of Summer Sessions, supplement those given in our last issue.

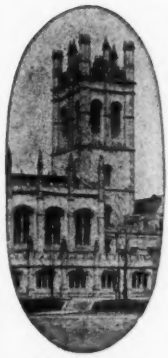
At *Oberlin College*, Professor W. D. CAIRNS will give courses in Freshman mathematics in the summer session, June 20 to August 7.

The Faculty of Applied Science of *Queen's University*, Kingston, will hold this year a special summer session for returned soldiers. The date of opening, about May 1, depends somewhat upon the time of demobilization, and the session will continue until the regular session begins in October. As a great many students enlisted early in the war, before they had finished their year, this session will serve either as a "refresher" course or will count as a year towards a degree.

University of Kansas, June 17–July 25, 1919. By Professor C. H. ASHTON: Differential Calculus (3 semester hrs. credit), and Advance analytic geometry (3 hrs.).—By Professor U. G. MITCHELL: Series (3 hrs.) and Teachers' course (3 hrs.). By Professor J. J. WHEELER: College algebra (2 hrs.) and Analytic geometry (4 hrs.). Second session, July 28–August 22. By Professor E. B. STOUFFER: Trigonometry (2 hrs.) and Theory of equations (2 hrs.).

University of Wisconsin: By Professor C. S. SLICHTER, Fourier series, 1 credit; Mechanics, 2 credits; Algebra, 2 credits. By Professor L. W. DOWLING: Equations of the third and higher degrees, 1 credit; Projective geometry, 2 credits; Analytic geometry, 2 credits. By Professor W. W. HART: The content of elementary mathematics, 1 credit; The teaching of secondary mathematics, 2 credits; Plane trigonometry and logarithms, 2 credits. By Professor H. C. WOLFF: Practical computation, 2 credits; Definite integrals, 1 credit; Calculus, 2 credits. By Dr. C. P. PAINE: Integral calculus, 4 credits. By Mr. R. W. BABCOCK: Elementary analysis, 4 credits; Solid geometry, 2 credits. A course in surveying will be given in the College of Engineering.

University of Chicago, June 16–August 29, 1919. By Professor G. A. BLISS: Differential equations (Lie theory), 4 hours; Differential calculus, 5 hours. By Professor W. D. MACMILLAN: Celestial mechanics, 4 hours. By Professor F. R. MOULTON: The solution of numerical differential equations, 4 hours. By Professor H. E. SLAUGHT: Elliptic integrals, 4 hours; Integral calculus, 5 hours. By Professor E. J. WILCZYNSKI: Metric differential geometry, 4 hours; College algebra, 5 hours. By Professor J. W. A. YOUNG: Solid analytic geometry, 4 hours; Plane trigonometry, 5 hours. By Professor A. B. COBLE (University of Illinois): Elliptic modular functions, 4 hours; Plane analytic geometry, 5 hours. By Professor T. H. HILDEBRANDT (University of Michigan): Theory of functions of a real variable, 4 hours; Limits and series, 5 hours. By Professor G. W. MYERS (School of Education): The teaching of secondary mathematics, 5 hours.



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Registrar, University, Madison, Wisconsin

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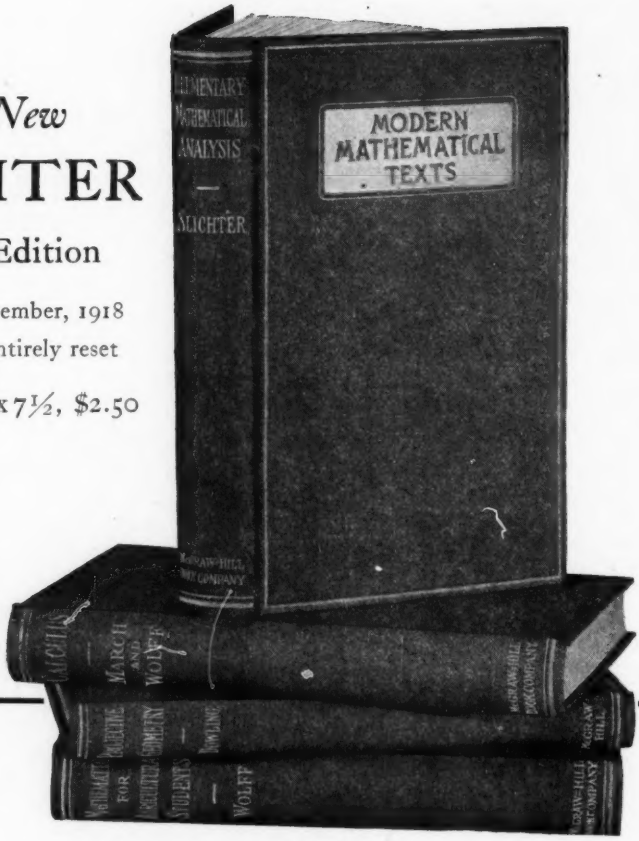

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THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual and institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

HOW THE MAP-PROBLEM WAS SOLVED IN THE WAR.[1]

By KURT LAVES, University of Chicago.

Introductory Considerations which Led to the Selection of the Lambert Projection for the War Maps of Northern France.

For the pre-war maps of northern France the Bonne projection was used, with the Meridian of Paris as primary meridian. The Bonne projection (Rigobert Bonne, 1727–1795) is a modified conical projection. As an "equivalent" projection any area on the map is equal to the corresponding area on the Earth's surface. For all peaceful occupations this is an essential requirement, while for the purposes of war, in particular those of heavy artillery, the essential requirement is that the angles and distances radiating from *any point* of the territory mapped should with only inappreciable errors be obtainable from the data furnished by the map. The theory of the Bonne projection will show that the distortions in angle and distance increase with the longitude from the primary meridian. At the eastern frontier of northern France the angular errors[2] amount to 18′ and the linear errors to 1 : 379. Such errors are not negligible in artillery fire. The eventual subsequent transfer of the battlefront in an easterly direction for which the military command had to make early provision in their map-department would have necessitated the introduction of a second and perhaps of a third and fourth primary meridian under the Bonne projection, in order that these errors might be kept within specified limits. Two, and eventually three or even more, sets of maps which furnish *different* data for the territory covered by them in common are a source of confusion which may lead to disastrous results. This fact, coupled with the relatively large distortions in angle and distance at the edges of each set of maps mentioned above counted heavily against the retention of the Bonne projection. The modern fire-direction of a battery requires the determination of distance and azimuth from the rectangular coördinates of target and battery. The kilometer-grid printed on the map permits of a rapid solution of this problem. The meridians on the Bonne projection are sinusoidal curves and "convergence of meridians," *i. e.*, the angle between the meridian of a point P and that of the primary meridian, is not readily obtainable by calculation since it is a function of both latitude and longitude of P. The determination of "Bonne-North" (corresponding to Lambert-North, *i. e.*, direction of primary meridian at a point) would have proved a rather lengthy and knotty process for the average battery officer even with the help of auxiliary tables.

[1] Read at the joint meeting of the Mathematical Association of America and of the American Mathematical Society at Chicago, December 27, 1918.

[2] Walton C. Clark, *Heavy Artillery Orientation*, p. 33; published by Coast Artillery School, Ft. Monroe, Va. $0.75.